A geometric approach to factor model identification

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Abstract

We use the geometric representation of factor models to represent the factor loading structure by sets corresponding to unit-specific non-zero loadings. We formulate global and local, rotational identification conditions based on set conditions. We propose two algorithms to efficiently evaluate Sato (1992)'s counting rule. We demonstrate the efficiency and the performance of the algorithms by a simulation study. An application to exchange rate returns illustrates the approach.

Keywords: Factor model, Global and local identification, Set representation.

JEL: C10, C52, C55, E44, F31

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1 Introduction

Identification in factor modelling often relates to identifying the scale of factors and the orientation of the factor basis determining factor order and sign. The more fundamental issue of whether the decomposition into common and idiosyncratic variation is (uniquely) identified has eclipsed, as increasingly high-dimensional data sets have become available (N large), and, relying on their informativeness, few factors (K small) are usually expected to capture the bulk of data covariation. In exploratory analysis, standard non-parametric methods extract factors by principal or frequency components analysis (Stock and Watson, 2002; Forni et al., 2005). While in these approaches factors are determined and ordered according to the average share of covariation explained across series, the interpretation of factors gets increasingly blurred when they are extracted from high-dimensional sets of heterogeneous data. The situation is similar when factor extraction is based on singular value decomposition (Hoff, 2007; Chan et al., 2018). In the parametric framework, factor identification and ordering is usually induced by imposing restrictions on the factor loading matrix. Widely used approaches impose restrictions before estimation (Geweke and Zhou, 1996; Aguilar and West, 2000; Bernanke et al., 2005) while more recent ones apply efficient and order-invariant estimation, where factor identification and ordering is obtained by post-processing (Aßmann et al., 2016; Chan et al., 2018; Kaufmann and Schumacher, 2019).

Originating in psychometrics, a primary goal in confirmatory factor analysis is to obtain, besides factor identification, a factor interpretation. This is induced either by rotating factors to maximize the share of factor-specific variation in specific units, see e.g. Thurstone (1935), Cattell (1978) or Bollen (1989), or imposing a simple structure on the factor loading matrix to determine factor interpretation, i.e. imposing zero loadings on series not expected to determine or be loaded by specific factors, see e.g. Jöreskog (1969), Jennrich (1978), Jöreskog (1979), Millsap (2001) or Mulaik (2010). These approaches have lately been applied also in econometrics (Bai and Ng, 2013; Kose et al., 2003; Moench et al., 2013). Sparse (exploratory) factor analysis draws on the advantage of inducing a simple structure into the factor loading matrix to extract relevant factors or eventually obtain factor interpretation (West, 2003; Lucas et al., 2006; Beyeler and Kaufmann, 2021). Recent advances in econometrics propose estimation procedures also relying on sparse principal components (Zou et al., 2006; Despois and Doz, 2023) or regularized estimation (Freyaldenhoven, 2023).

Identification has been discussed since the very early stages of factor modelling (Ledermann, 1937; Anderson and Rubin, 1956). Global identification dealt with the question of unique identification of the variance decomposition, related to the number of factors identifiable from the covariance structure of the data (Shapiro, 1982). Imposing a simple structure in confirmatory factor analysis raised similar issues (Millsap, 2001; Sato, 1992). Nowadays, sparse exploratory factor analysis in high-dimensional data settings raises the issue of whether a sparse reduced rank representation is (uniquely) identifiable from the data covariance (Frühwirth-Schnatter et al., 2023; Kaufmann and Pape, 2023), given that in a factor model of potentially increasing dimension the factor loading matrix may become increasingly sparse.

This raises anew the issue of the identification of the variance decomposition (global identification) and factors (rotational identification), which motivates the present paper. We review relevant global and local identification and uniqueness conditions that have been documented in the literature. Besides conditions derived in Anderson and Rubin (1956) and Sato (1992) related to global identification, we will review conditions for mode or rotational identification (Bekker, 1986; Neudecker, 1990; Bai and Wang, 2014). Our contribution relies on the geometric representation of factor models (Lawley and Maxwell, 1971) to derive a geometric approach to identification. Full-rank, rotational identification is based on set identification conditions, where only (at least) K instead of $2^K - 1$ sets of subspaces lying in the K-dimensional factor space need to be populated for full-rank, rotational identification. Sato (1992) derived a counting rule which provides a necessary condition for global identification of the reduced-rank variance decomposition. Checking the rule needs an exponentially growing combinatorial number of evaluations $(2^K - 1)$, which becomes quickly computationally prohibitive for factor models of increasing dimensions. Based on our geometric approach to identification, we provide two algorithms, the $\mathcal{O}(K^2)$ and the set-based algorithm for evaluating efficiently Sato's counting rule.

Related to our paper, Frühwirth-Schnatter et al. (2023) present a thorough analysis of identification based on the so-called generalized lower triangular (GLT) structure, where the leading non-zero loadings of factors (pivot elements) need to be in different rows (pivot rows). While imposing or rotating into a GLT structure may serve the evaluation of model identification, we will argue that imposing a GLT structure for estimation is prone to the same problem as the widely applied positive lower triangular (PLT) identification constraints. Imposing a GLT structure is not order-invariant, may induce an orientation of the factor basis destroying or biasing the sparse structure underlying the data. Our approach to identification suggests that requiring the loadings leading the pivot elements to equal zero is potentially misleading when the purpose is to estimate a sparse factor loading matrix. Rather, it suffices to determine pivot rows or *pivot series*, who when re-ordered first in the data set would lead to an unordered GLT structure in the factor loading matrix. The result of our set identification procedure can help in determining those pivot series.

In the next section, we introduce the factor model and briefly review published results and issues related to global and local identification, relevant for motivating our contribution. Section 3 relies on the geometric interpretation of factor models to introduce the geometric approach to identification. Based on these considerations, we propose two algorithms, the $\mathcal{O}(K^2)$ and the set-based algorithm, which allow an efficient evaluation of Sato's counting rule (Sato, 1992). Section 4 documents the efficiency and the performance of the algorithms by comparing them to the plain evaluation of Sato's rule. An application to exchange rate returns illustrates the approach. Section 5 concludes.

2 Representation and identification

We review results concerning global and local identification of factor models published in the literature, which are relevant for motivating and introducing the geometric approach

to identification.

2.1 Representation

Collect observed data in vector $Y = (y_1', \ldots, y_T')'$, where each repeated measurement y_t, $t = 1, \ldots, T$, denotes an $N \times 1$ vector of variables or units y_{it}, $i = 1, \ldots, N$, and can be represented as

$$\begin{aligned} &y_t = \Lambda f_t + \epsilon_t, \\ &E\left(f_t f_t'\right) = I_K,\ E\left(\epsilon_t \epsilon_t'\right) = \Sigma_\epsilon,\ \Sigma_\epsilon \text{ diagonal}, \end{aligned} \tag{1}$$

with $K << N$ and where f_t is a $K \times 1$ vector of latent factors, $\Lambda = \{\lambda_{ik} | i = 1, \ldots, N, k = 1, \ldots, K\}$ is the $N \times K$ factor loading matrix and ϵ_t is an $N \times 1$ vector of idiosyncratic components.[1] As common variation is captured by the factor component only, Σ_ϵ is diagonal and $E(f_t \epsilon_t') = 0$. Finally, identification issues we discuss in the following are independent of distributional assumptions; therefore, we do not specify any so far. We assume that first and second (unconditional) moments are, respectively, zero and constant, which means that observed data in (1) follows a covariance-stationary process.

Underlying factors are usually unobserved and have to be extracted from observed data variation, $\Sigma_y = E(y_t y_t')$:

$$\Sigma_y = \Lambda \Lambda' + \Sigma_\epsilon. \tag{2}$$

Finding a solution to (2) does not only mean mathematically solving the system of $N(N+1)/2$ independent equations. A valid decomposition requires Σ_ϵ to be positive definite and $\Sigma_y - \Sigma_\epsilon$ positive semi-definite and of lower-rank K.

Questions that arise are (a) does a solution exist and is it unique, which concerns *global* identification; (b) is Σ_ϵ unique, which concerns *local* identification, and (c) for an identified solution, how to determine the orientation of the factor basis which includes factor order and sign, which concerns *rotational* or *mode* identification. We briefly review results concerning the first two issues before elaborating in more details on the last one.

2.2 Global and local identification

The most general concept of identification implies that for the decomposition into Λ and Σ_ϵ, there exists only a single solution. Several results have established the necessary and sufficient conditions for identifying a factor model, some of which are discussed below. However, a complete solution to the global identification problem has not yet been achieved, see e.g. Millsap (2011).

[1] We assume without loss of generality an identity covariance matrix for factors, given that correlated factors $\tilde{f}_t$ can be de-correlated by using e.g. a Cholesky decomposition of the factor covariance: $E\left(\tilde{f}_t \tilde{f}_t'\right) = \Sigma_{\tilde{f}} = LL'$; $L^{-1}\Sigma_{\tilde{f}} L^{-1'} = I_K$. When post-multiplying $\tilde{\Lambda}$ with L, the factor model with correlated factors is observationally equivalent to system (1). $y_t = \tilde{\Lambda} L L^{-1} \tilde{f}_t + \epsilon_t = \Lambda f_t + \epsilon_t$.

Definition 2.1. *Global identification*
Model (2) is *globally identified*, if there is a Σ_ϵ with non-negative diagonal elements which solves (2) and yields a common covariance matrix $\Sigma_y - \Sigma_\epsilon$ that is positive-semidefinite and of reduced rank K (Anderson and Rubin, 1956, AR56). $\diamond$

Definition 2.2. *Local uniqueness*
Model (2) is *locally unique*, if in the neighbourhood of Σ_ϵ there is no other idiosyncratic variance matrix $\tilde{\Sigma}_\epsilon$ that solves (2) and yields a common covariance $\Sigma_y - \tilde{\Sigma}_\epsilon$ that is positive-definite and of reduced rank K (Shapiro, 1985, 1989). $\diamond$

A sufficient condition for global identification is given in Theorem 5.1 of AR56 (adjusted here in notation)

> *A sufficient condition for identification of Σ_ϵ and Λ up to multiplication on the right by an orthogonal matrix is that if any row of Λ is deleted there remain two disjoint submatrices of rank K.*

Further theorems provide either necessary or necessary and sufficient conditions for global identification of particular models, see also the overview in Bollen (1989, ch. 7).

Definition 1 includes stronger conditions than the criterion given by Ledermann's bound (Ledermann, 1937), which equates the number of equations with the number of unknowns in (2) and yields $\varphi(N) = (2N+1-\sqrt{8N+1})/2$. The bound yields a hypothetical criterion about the number of factors needed to obtain a reduced-rank decomposition as in (2). It is not very useful, though, because an algebraic solution does not exclude cases where Σ_ϵ contains negative elements (Heywood case) or $\Sigma_y - \Sigma_\epsilon$ is not positive-semidefinite.[2]

As regards the Ledermann bound, Shapiro (1982) showed that it is almost surely a lower bound for the number of factors needed to obtain a reduced rank common covariance.[3]

[2] A noteworthy insight from counting is that at least three loadings must be non-zero to identify one factor, as a minimum of six covariances is needed to identify six unknown parameters:

$$\Sigma_y = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 & \sigma_1\sigma_3 \\ \sigma_2\sigma_1 & \sigma_2^2 & \sigma_2\sigma_3 \\ \sigma_3\sigma_1 & \sigma_3\sigma_2 & \sigma_3^2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2+\sigma_{\epsilon 1}^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2+\sigma_{\epsilon 2}^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2+\sigma_{\epsilon 3}^2 \end{bmatrix}.$$

Theorem 4.2 of AR56 states additional conditions for a solution satisfying a factor representation (adjusted in notation)

> *A necessary and sufficient condition that Σ_y be a covariance matrix of a factor analysis model with one factor is that $N(N-1)/2 - N$ independent tetrad conditions are satisfied and*
>
> $$0 \leq \frac{\sigma_{ki}\sigma_{ij}}{\sigma_{kj}} \leq \sigma_i^2, \ \sigma_{kj} \neq 0$$
>
> *for one pair $(j \neq k)$ for each i.*

When $N = 3$, no tetrad conditions are left and the solution is exact: $\sigma_{\epsilon,k} = \sigma_k^2 - \lambda_k^2$, $k = 1, \ldots, K$. When $N = 4$, two additional tetrad conditions need to be fulfilled: $\sigma_{31}\sigma_{42} - \sigma_{41}\sigma_{32} = 0$, $\sigma_{21}\sigma_{34} - \sigma_{31}\sigma_{24} = 0$. For $K > 1$ conditions become more complicated (Wilson and Worcester, 1939, AR56). From these considerations, we conclude that whether a representation (2) exists depends ultimately on characteristics of the data covariance Σ_y (AR56, last paragraph of Section 5).

[3] That is, there are "almost no" sample covariance matrices that have factor structure with the number of factors lower than the Ledermann bound.

Shapiro (1985) further showed that any Σ_ϵ which solves (2) is almost surely non-unique if K is above the Ledermann bound and almost surely locally unique when K is at or below the Ledermann bound.

Note that AR56 do not discuss the Ledermann bound explicitly, but evaluate the number of degrees of freedom of the (static) factor model, which is

$$c = \frac{N(N+1)}{2} - N - \left(NK - \frac{K(K-1)}{2}\right) = \frac{(N-K)^2 - N - K}{2}. \tag{3}$$

They state that an algebraic solution is possible when $c \leq 0$ whereas when $c > 0$, there is generally no solution. In the case of $c \leq 0$, the solution may not be unique. Hence, it may be appropriate to increase c, e.g. by imposing additional restrictions on Λ. Both confirmatory factor analysis and sparse factor analysis increase c, albeit usually not in a way that results in a unique solution. An increase in the number of factors K, on the other hand, reduces c. In practical context, the case $c > 0$ seems much more relevant. As it is almost impossible to find a parsimonious solution for a factor representation of a given sample covariance matrix S_y, see Shapiro (1982), it is usually assumed that $S_y \neq \Sigma_y$, and $\hat{\Sigma}_y = \hat{\Lambda}\hat{\Lambda}' + \hat{\Sigma}_\epsilon$. Implicitly, $\hat{\Sigma}_y$ thus "satisfies some c conditions" (AR56), imposed through the structure of the factor model. Whether $\hat{\Sigma}_y$ is a plausible estimate, however, also depends on the type of model used. For instance, a sparse factor model with many factors is much more flexible than a dense model with fewer factors, even if the number of parameters is the same.

Sato (1992) frames global identification into the analysis of correlation, and represents regions of unique decompositions geometrically. He qualifies and extends results of AR56, in particular derives a necessary condition for Theorem 5.1 of AR56 to hold (Sato, 1992, Theorem 3.3., adjusted here in notation):

> *A necessary condition for satisfying the condition of Theorem [5.1 in AR56] is that the submatrices which consist of each q columns of ΛG have at least $(2q+1)$ nonzero rows for every nonsingular G $(q = 1, 2, \ldots, K)$.*

This "counting rule" (Frühwirth-Schnatter et al., 2023) encompasses Theorem 5.6 in AR56, which states a necessary condition for global uniqueness to be that each column of ΛG (for every non-singular matrix G) should have at least three non-zero loadings. The results prove useful to assess global and local identification in $K << N$ settings, where the factor loading matrix may be sparse (West, 2003).

2.3 Rotational or mode identification

Definition 2.3. *Equivalence class*
A locally unique model defines an *equivalence class*. The equivalence class contains all solutions for which $\Sigma_y - \Sigma_\epsilon = \Lambda\Lambda'$ is positive-semidefinite and has reduced rank K, see also Millsap (2001). ◇

Definition 2.4. *Mode identification*
A solution in an equivalence class which identifies the orientation and scale of the factor basis, factor position and sign, is called a *mode-identified* solution. ◇

Assuming a locally unique solution to (2) is given, it is clear that (2) is identified up to rotation, given that $\Lambda\Lambda'$ can always be replaced by $\Lambda QQ'\Lambda'$ using an orthogonal matrix Q, or, if $\Sigma_{\tilde{f}} \neq I_K$, $\tilde{\Lambda}\Sigma_{\tilde{f}}\tilde{\Lambda}'$ can always be replaced by $\Lambda^*\Sigma_{f^*}\Lambda^{*\prime} = \tilde{\Lambda}QQ^{-1}\Sigma_{\tilde{f}}Q^{-1\prime}Q'\tilde{\Lambda}'$ using an arbitrary, non-singular matrix Q.

A first set of restrictions fixes the basis and the scale of the factor space. An obvious assumption is to set $\Sigma_f = I_K$, which defines an orthogonal basis of unit scale. This provides $K(K+1)/2$ restrictions. Further, assuming that every rotation $Q^{-1}\Sigma_f Q^{-1\prime} = I_K$ should not induce a change in the assumption $\Sigma_f = I$ reduces the set of rotation matrices Q to orthogonal matrices, of which $K(K-1)/2$ elements are free. Therefore, to fix the orientation of the basis, we need an additional $K(K-1)/2$ restrictions.

AR56 suggest three possible sets of restrictions to obtain rotational identification, the first two of which are typically used in exploratory factor analysis. The first set of constraints restricts the $K(K-1)/2$ upper diagonal elements of Λ to zero, i.e. $\lambda_{ik} = 0$ for $k > i$ (Geweke and Singleton, 1981; Geweke and Zhou, 1996). The position of the zeros additionally defines factor order. The first factor hence loads only on variable 1, while the second loads on variable 1 and 2 and so on. Factor sign is identified by requiring $\lambda_{kk} > 0$ for $k = 1, \ldots, K$, i.e. diagonal elements to be positive. The second set of constraints restricts Λ to satisfy $\Lambda'\Lambda = D$ with D diagonal (Stock and Watson, 2002). In this case, factor order is determined by requiring that the nonzero diagonal elements in D be distinct and ordered in increasing or decreasing order of magnitude. The last set of constraints requires that $\Lambda'\Sigma_\epsilon^{-1}\Lambda$ is a diagonal matrix, whose nonzero diagonal elements are distinct and ordered in increasing or decreasing order of magnitude.

Other identification constraints assume Σ_f diagonal, albeit without unit scaling. Then, in addition to requiring $\lambda_{ik} = 0$ for $k > i$, the elements $\lambda_{kk} = 1$ for $k = 1, \ldots, K$ are set to unity. This setup is used in e.g. West et al. (2001). Alternatively, Σ_f may be left altogether unconstrained, which yields an unscaled oblique factor basis. For rotational identification we then need K^2 restrictions. Constraining the leading $K \times K$ submatrix of Λ, $\Lambda_K = I_K$ represents a possibility to achieve rotational identification, as suggested in Jöreskog (1979) and applied in Bernanke et al. (2005). In this case, the first K variables are the factors, i.e. they define factor position and factor scale. The restrictions imposed on the leading K rows of Λ may alternatively also be spread across any subset of K rows in Λ. Imposing them on the leading K rows is just a matter of convenience. However, even though these constraints guarantee mode identification, they are not likelihood invariant with respect to variable ordering and may lead to serious estimation issues, see e.g. Millsap (2001), Chan et al. (2018). An illustration is given in Subsection 3. Recent advances therefore suggest to estimate just-identified or unrestricted sparse factor models, and recover mode identification by processing the estimation output (Aßmann et al., 2016; Kaufmann and Schumacher, 2019; Despois and Doz, 2023).

Mode-identification in a dynamic factor model[4] needs essentially the same K^2 number of restrictions to identify the basis, scale and orientation of the factor space. As in the static case, identification of factor position and sign requires additional restrictions. The analogue to setting $\Sigma_f = I$ is to set the covariance matrix of the factor innovations to the identity matrix, $\Sigma_\eta = I$. This renders factors conditionally independent. Although dynamic factor processes are assumed to be diagonal in some applications (Kose et al., 2003; Otrok and Whiteman, 1998), this does not have to be imposed generally in a $K << N$ (sparse) environment (Beyeler and Kaufmann, 2021). A minimum of $K(K-1)/2$ zero restrictions on Λ and a specific sparse structure are necessary for mode identification, see below.

Discussions in confirmatory factor analysis have shown that imposing additional constraints on the factor loading matrix may result in specifications that lack unique mode identification (Jennrich, 1978).[5] These issues translate to high-dimensional settings in confirmatory and exploratory factor analysis as well, where Λ may be sparse or subject to more zero restrictions than necessary for mode identification. In particular, some results show that multiple sparse structures may be the rule rather than the exception. Algina (1980) provided an explanation for the counterfactual example published in Jennrich (1978) and derived first rules concerning mode identification based on a rank condition. Bekker (1986) and Neudecker (1990) expressed the set of constraints placed on the factor loadings matrix as $R\text{vec}(\Lambda) = r$. Bekker (1986) showed that a necessary and sufficient

[4]Parametric extensions of (1) may include dynamic processes for f_t and ϵ_t, $f_t = \phi_1 f_{t-1} + \cdots + \phi_p f_{t-p} + \eta_t$ and $\epsilon_{it} = \psi_1 \epsilon_{i,t-1} + \cdots + \psi_q \epsilon_{i,t-q} + \upsilon_{it}$, respectively.

[5] For example, Λ, although identified when re-ordering time series, can be transformed to ΛC (again identified when re-ordering time series), with a different loading structure below the identical first three rows of loadings (Householder transformation):

$$\Lambda = \begin{bmatrix} 1.41 & 0 & 1.12 & 0 \\ 1.42 & -0.79 & 0 & 0 \\ 0.67 & 0.89 & -0.10 & -1.09 \\ -1.21 & 0 & 0 & 0.03 \\ 0.72 & -1.07 & 0.32 & -1.21 \\ 1.63 & -0.81 & 0.31 & 0 \\ 0.81 & -0.34 & 0.23 & -0.41 \\ 1.03 & 0 & 0 & 0 \\ 0.73 & 0.33 & -0.16 & -0.77 \\ -0.30 & -0.75 & 0.63 & 0.37 \end{bmatrix} \rightarrow \Lambda C = \begin{bmatrix} 1.41 & 0 & 1.12 & 0 \\ 1.42 & -0.79 & 0 & 0 \\ 0.67 & 0.89 & -0.10 & -1.09 \\ -0.99 & 0.38 & -0.27 & 0.50 \\ 1.52 & 0.37 & -0.69 & 0.55 \\ 1.67 & -0.73 & 0.26 & 0.09 \\ 0.99 & 0 & 0 & 0 \\ 0.84 & -0.35 & 0.24 & -0.43 \\ 0.76 & 0.38 & -0.21 & -0.70 \\ 0 & -0.20 & 0.24 & 1.05 \end{bmatrix},$$

where $C = I_K - 2vv'$, $v = \text{null}(\Lambda^*)$ and Λ^* collecting the first rows of Λ determined by $\min(i_{max}, k_{max}-1)$, where i_{max} and k_{max} refer to the largest row and column with a non-zero element on the upper diagonal of Λ (Here $i_{max} = 4$, $k_{max} - 1 = 3$), and

$$v = null\left(\begin{bmatrix} 1.41 & 0 & 1.12 & 0 \\ 1.42 & -0.79 & 0 & 0 \\ 0.67 & 0.89 & -0.10 & -1.09 \end{bmatrix}\right) \rightarrow C = \begin{bmatrix} 0.81 & -0.34 & 0.24 & -0.41 \\ -0.34 & 0.39 & 0.42 & -0.74 \\ 0.24 & 0.42 & 0.70 & 0.52 \\ -0.41 & -0.74 & 0.52 & 0.09 \end{bmatrix}.$$

condition for mode identification requires that[6]

$$\text{rk}\begin{pmatrix} D_K^+ \\ R(I_K \otimes \Lambda) \end{pmatrix} = K^2,$$

and Neudecker (1990) proved that

$$\text{rk}\begin{pmatrix} D_N^+(\Lambda \otimes I_N) \\ R \end{pmatrix} = NK$$

is equivalent to Bekker's condition under the assumption $\text{rk}(\Lambda) = K$. More recently, Bai and Wang (2014) extend these results to mode identification conditions for dynamic factor models, $\Phi(L)f_t = B(L)\eta_t$, where $\Phi(L)$, $B(L)$ represent polynomials in the lag operator. By defining

$$\Phi = [\Phi_1, \ldots, \Phi_p], \tilde{\Phi} = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_p \end{bmatrix}, B = [B_1, \ldots, B_q], \tilde{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix},$$

and expressing restrictions as $R_\Lambda \text{vec}(\Lambda) = r_\Lambda$, $R_\Phi \text{vec}(\Phi) = r_\Phi$, $R_B \text{vec}(B) = r_B$, the rank condition becomes:

$$\text{rk}\begin{pmatrix} D_K^+ \\ R_\Lambda (I_K \otimes \Lambda) \\ R_\Phi \left[I_K \otimes \tilde{\Phi} - \Phi' \otimes I_K\right] \\ R_B \left[I_K \otimes \tilde{B} - B' \otimes I_k\right] \end{pmatrix} = K^2.$$

Importantly, these rank conditions do not guard against singleton or spurious factor loadings (only one non-zero factor loading in a column), leading to an indeterminacy between factor and idiosyncratic variance.[7] Hence, fulfilled rank conditions do not imply global identification. Global identification is assumed to hold in a first place.

3 A geometric approach to identification

3.1 Full-rank mode identification: Set conditions

Rank conditions can be derived based on a geometric interpretation of factor models, where Σ_f spans a possibly correlated factor basis[8] and each row λ_i in Λ represents weights

[6] The matrix D_K^+ is the Moore-Penrose inverse of the duplication matrix D_K, which transforms the half-vectorization of a $K \times K$ symmetric matrix X into its vectorization as $D_K \text{vech}(X) = \text{vec}(X)$.

[7] Assume that factor K has a nonzero loading only for variable i. Then the variance for variable i can be decomposed as

$$\sum_{k=1}^{K-1} \lambda_{ik}^2 \sigma_{fk}^2 + \underbrace{\lambda_{iK}^2}_{\equiv 1} \sigma_{fK}^2 + \sigma_i^2 = \sum_{k=1}^{K-1} \lambda_{ik}^2 \sigma_{fk}^2 + (\sigma_{fK}^2 + c) + (\sigma_i^2 - c)$$

for $c \in (-\sigma_{fK}^2, \sigma_i^2]$. The case $c = -\sigma_{fK}^2$ is inconsistent with the unity constraint and would imply that factor K can be omitted from the analysis. Conversely, the case $c = \sigma_i^2$, implies that all remaining variable i's variation unexplained by the first $K-1$ factors is assigned to factor K.

[8] $\Sigma_f = I_K$ corresponds to an orthonormal factor basis.

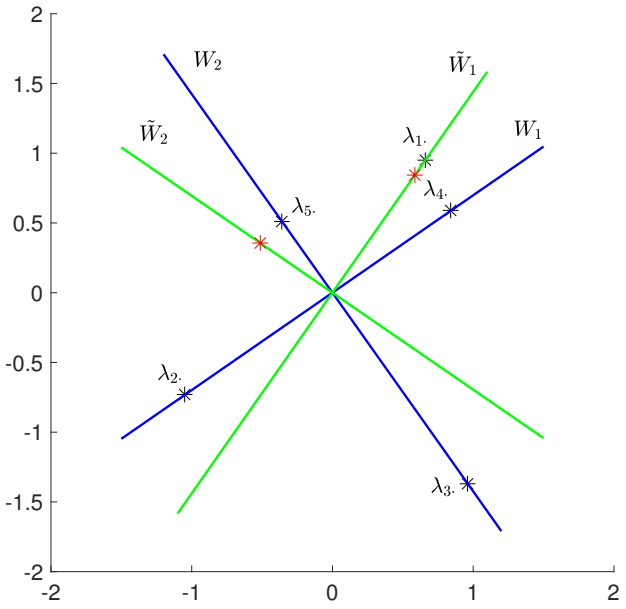


Figure 1: Coordinates representing five rows of two-dimensional factor loadings, of which each of two pairs are located in a 1-dimensional subspace (blue lines). The green lines correspond to a factor basis induced by a rotation into a GLT structure. The red stars indicate factor loadings when additional moderate shrinkage is induced into the GLT structure.

attached to basis vectors and corresponds to cartesian coordinates in a K-dimensional space (Lawley and Maxwell, 1971). It is useful to introduce some geometric and topological concepts. Whereas the considered concepts can generally be defined for various fields, we are only interested in real numbers, see e.g. Boothby (2002) and James (1976). Denote as a K-frame a set of K independent column vectors in the $\mathbb{R}^N$ with $K < N$, or, as an $N \times K$ matrix with full column rank. The set of all K-frames in the $\mathbb{R}^N$ is then denoted as the (real) non-compact Stiefel manifold $V(K, N)$.[9] As the K independent column vectors in a K-frame span the K-dimensional (real) vector space $\mathbb{R}^K$, we consider its k-dimensional subspaces $k < K$. The set of all k-dimensional linear subspaces of $\mathbb{R}^K$ defines the (real) Grassmann manifold $Gr(k, K)$. For instance, $Gr(1, 2)$ is the set of all lines through the origin in a plane. Finally, the set of all orthogonal $K \times K$ matrices is denoted as the (real) orthogonal group $O(K)$, corresponding to an orthogonal factor basis.

A necessary condition for the column space of Λ to exist is that Λ has full column rank K, such that a subset of rows of Λ, say Λ_a, exists that spans the $\mathbb{R}^K$. Equivalently, defining the vector space W that contains all row vectors of Λ (the K-frame defined above), i.e. $W = \{\lambda_{i\cdot} | i = 1, \ldots, N\}$, it must hold that $W \notin Gr(k, K)$ for $k < K$.

[9]Note that we do not require the columns of the K-frame to be orthogonal, as sometimes defined.

For example,[10] Figure 1 plots the following loading matrices as coordinates:

$$\Lambda = \begin{pmatrix} \Lambda_a \\ \hline \Lambda_b \end{pmatrix} = \left(\begin{array}{rr} 0.66 & 0.95 \\ -1.05 & -0.73 \\ \hline 0.96 & -1.37 \\ 0.84 & 0.59 \\ -0.36 & 0.51 \end{array}\right), \; \tilde{\Lambda} = \left(\begin{array}{rr} 1.09 & 0.40 \\ -1.28 & 0.00 \\ \hline 0.00 & -1.68 \\ 1.02 & 0.00 \\ 0.00 & 0.63 \end{array}\right), \; \Lambda^{GLT} = \left(\begin{array}{rr} 1.16 & 0.00 \\ -1.20 & 0.45 \\ \hline -0.58 & -1.57 \\ 0.96 & -0.35 \\ 0.21 & 0.59 \end{array}\right), \tag{4}$$

where coordinates for Λ are specified in terms of the x- and y-axis. Two pairs of row vectors in Λ, are each located in a 1-dimensional subspace, $\{W_1, W_2\} \in Gr(1,2)$ for $W_1 = \{\lambda_{2\cdot}, \lambda_{4\cdot}\}$ and $W_2 = \{\lambda_{3\cdot}, \lambda_{5\cdot}\}$. The subspaces span an orthogonal factor basis $W_1 \perp W_2$, indicated with blue lines. The sparse loading matrix $\tilde{\Lambda}$ corresponds to the rotated factor basis. The example also illustrates the importance of choosing units when setting pre-defined identification restrictions onto the factor loading matrix. Choosing either $\lambda_{2\cdot}$ and $\lambda_{4\cdot}$ or $\lambda_{3\cdot}$ and $\lambda_{5\cdot}$ as leading units Λ_a in Λ combined with identification restrictions such as lower diagonal or diagonal, would fail in identifying a second factor as each set of units is loaded by a single factor only. On the other hand, using the first two rows of Λ to rotate into a (generalized) lower triangular structure (Frühwirth-Schnatter et al., 2023) destroys the sparse structure of the loading matrix.[11] The green lines reflect the rotated factor basis corresponding to Λ^{GLT} in which the factor loading structure has been rotated into a lower triangular structure. These considerations motivate to base inference on order-invariant estimation and identify factors, including their order and sign (rotational identification), after estimation by processing the posterior output (Aßmann et al., 2016; Chan et al., 2018; Kaufmann and Schumacher, 2019).

Definition 3.1. *Subset enumeration (single index)*
Consider the set $\mathcal{I} = \{1, \ldots, K\}$ with $K \geq 1$. The power set $\mathcal{P}(\mathcal{I})$ contains all 2^K subsets of $\mathcal{I}$, which can be enumerated by $\mathcal{I}_w$, where $w = \sum_{i=1}^{K} 2^{i-1} 1_{\{i \in \mathcal{I}_w\}}$, and the empty set is $\mathcal{I}_0 = \{\}$. $\diamond$

Definition 3.2. *Subset enumeration (double index)*
Consider the set $\mathcal{I} = \{1, \ldots, K\}$ with $K \geq 1$. The power set $\mathcal{P}(\mathcal{I})$ contains all 2^K subsets of $\mathcal{I}$, with $\binom{K}{k}$ subsets of dimension k for $0 \leq k \leq K$. To distinguish between subsets

[10] We use the same numerical example as in Kaufmann and Pape (2023)

[11] Given the order non-invariance of the (generalized) lower triangular structure, inducing additional shrinkage for factor loadings below the leading non-zero loading of each factor when estimating the model might additionally bias the inference on the underlying factors and factor loading matrix. For example, inducing moderate shrinkage may pull $\lambda_{4\cdot}$ and $\lambda_{5\cdot}$ to lie on $\tilde{W}_1$ and $\tilde{W}_2$, respectively:

$$\Lambda^{GLT} = \left(\begin{array}{rr} 1.16 & 0.00 \\ -1.20 & 0.45 \\ \hline -0.58 & -1.57 \\ 0.96 & -0.35 \\ 0.21 & 0.59 \end{array}\right), \Lambda^{GLT,shrink} = \left(\begin{array}{rr} 1.16 & 0.00 \\ -1.20 & 0.45 \\ \hline -0.58 & -1.57 \\ 1.03 & 0.00 \\ 0.00 & 0.62 \end{array}\right), \tag{5}$$

see the coordinates in red in Figure 1. Obviously, this procedure blurs factor interpretation, biases the strong correlation across either pair of units 2 and 4 or 3 and 5, and overemphasizes the importance of Factors 1 and 2 for, respectively, units 4 and 5.

of different sizes, use index k to indicate the size of the subset and index l to enumerate the subsets of the same size.

For every $\mathcal{I}_{k,l} \subseteq \{1,\ldots,K\}$, $|\mathcal{I}_{k,l}| = k$ and $\psi(\mathcal{I}_{k,l}) = \sum_{i=1}^{K} 2^{K-i} 1_{\{i \in \mathcal{I}_{k,l}\}}$, such that $\psi(\mathcal{I}_{k,l}) > \psi(\mathcal{I}_{k,h})$ for every $l < h$. The empty set is $\mathcal{I}_{0,1} = \{\}$. $\diamond$

For instance, let $\mathcal{I} = \{1,2,3\}$. Then there exist three two-element subsets of $\mathcal{I}$, namely $\mathcal{I}_3 = \{1,2\}$ with (see Definition 3.1) $w = 2^{1-1} + 2^{2-1} = 1 + 2 = 3$, $\mathcal{I}_5 = \{1,3\}$ with $w = 2^{1-1} + 2^{3-1} = 1 + 4 = 5$ and $\mathcal{I}_6 = \{2,3\}$ with $w = 2^{2-1} + 2^{3-1} = 2 + 4 = 6$. Using Definition 3.2, we associate $\mathcal{I}_3 = \mathcal{I}_{2,1}$, $\mathcal{I}_5 = \mathcal{I}_{2,2}$ and $\mathcal{I}_6 = \mathcal{I}_{2,3}$, since $|\mathcal{I}_3| = |\mathcal{I}_5| = |\mathcal{I}_6| = 2$, and $\psi(\mathcal{I}_{2,1}) = 2^{3-1} + 2^{3-2} = 4 + 2 = 6$, $\psi(\mathcal{I}_{2,2}) = 2^{3-1} + 2^{3-3} = 4 + 1 = 5$, and $\psi(\mathcal{I}_{2,3}) = 2^{3-2} + 2^{3-3} = 2 + 1 = 3$ ($\psi(\mathcal{I}_{2,1}) > \psi(\mathcal{I}_{2,2}) > \psi(\mathcal{I}_{2,3})$).

Definition 3.3. *Subspace representation of an orthogonal factor model*
For a general subspace representation of an orthogonal factor model with K factors, let

$$W_k \in Gr(1,K),\ W_{k_i} \perp W_{k_j},\ k_i \neq k_j,\ k_i, k_j \in \{1,\ldots,K\},$$

and

$$W_{\mathcal{I}_{k,l}} = \bigcup_{j \in \mathcal{I}_{k,l}} W_j \quad \text{and} \quad W_{\mathcal{I}_w} = \bigcup_{j \in \mathcal{I}_w} W_j.$$

Note that for $\mathcal{I}_{K,1} = \mathcal{I}_w$ with $w = 2^K - 1$, $W_{\mathcal{I}_{K,1}} = \bigcup_{j=1}^{K} W_j$ spans the $\mathbb{R}^K$. $\diamond$

Recall from Equation (3) that Λ has $NK - \frac{K(K-1)}{2}$ distinctly identified parameters. Assume a dense Λ, mode-identified by a lower diagonal structure. We obtain $\lambda_{i\cdot} \in W_{\mathcal{I}_{i,1}}$ for $i \in \{1,\ldots,K-1\}$, and $\lambda_i \in W_{\mathcal{I}_{K,1}}$ for all $i \geq K$. The equivalence class can be obtained by $P_N \Lambda P_K H$, where P_N and P_K are N-dimensional row- and K-dimensional column-permutation matrices, respectively, and $H \in O(K)$ is a rotation matrix. Sparse loading matrices, on the other hand, imply $\lambda_i \in W_{\mathcal{I}_w}$, where $0 \leq |\mathcal{I}_w| < K$ for (almost) every $i \in \{1,\ldots,N\}$.

A special case obtains if $\{\lambda_i\}_{i=1}^N$ can be partitioned into K subsets where the k^{th} subset $\{\lambda_{i_k\cdot}\}_{i_k=1}^{N_k} \in W_k$. This is the congeneric factor model of Jöreskog (1971) with orthogonal factors. If the factors are allowed to be correlated, the constraint $W_{k_i} \perp W_{k_j}$ can be replaced by $W_{k_i} \neq W_{k_j}$ for all $k_i \neq k_j$, $k_i, k_j \in \{1,\ldots,K\}$, which renders the model a lot more flexible.

Using sets reflecting subspaces as defined in Definition 3.3, we can formulate set-based conditions to evaluate identification properties of factor models. We first assign each row vector $\{\lambda_i\}_{i=1}^N$ to the set $\mathcal{I}_w$ which corresponds uniquely to its non-zero loadings, $\lambda_i \in W_{\mathcal{I}_w}$.[12] We denote by N_w the number of rows $\lambda_{i\cdot}$ assigned to $\mathcal{I}_w$, $\{\lambda_{i_w\cdot}\}_{i_w=1}^{N_w} \in W_{\mathcal{I}_w}$.

[12] As we are interested in characterizing the sparse structure of λ, we assign each row to exactly one $\mathcal{I}_w$, the one that corresponds uniquely to the non-zero loadings. Note that generically, every $\lambda_{i\cdot} \in W_{\mathcal{I}_w}$ is also element of $W_{\mathcal{I}_x}$, for every $\mathcal{I}_x \supseteq \mathcal{I}_w$.

It turns out that full rank identification does not require all index sets to be non-empty. Rather, it suffices that at least K subsets be non-empty and span the K-dimensional space. Rotational identification can be based on K elements, each one taken from a different set. We formulate two conditions.

Definition 3.4. *Set identification*
For a (sparse) factor loading matrix Λ, assign each row $\lambda_{i\cdot}$ to the set $\mathcal{I}_w$ corresponding uniquely to the non-zero factor loadings, $\lambda_{i\cdot} \in W_{\mathcal{I}_w}$. Λ is set identified, if at least K sets are non-empty and the union over these sets spans the $\mathbb{R}^K$ space, $\bigcup_{w|N_w>0} \mathcal{I}_w = \{1, \ldots, K\}$.
$\diamond$

Condition 3.1. *Full rank set identification: A set identified (sparse) factor loading matrix Λ has full column rank, $\Lambda \in V(K, N)$, if K rows of factor loadings, $\lambda_{i_k\cdot}$, $k = 1, \ldots, K$, each assigned to a different set, $\mathcal{I}_{w_{i_k}} \neq \mathcal{I}_{w_{i_l}}$, $k \neq l$, are set identified and form a full-rank matrix.*

Condition 3.2. *Set-based mode-identification: A (sparse) factor loading matrix is mode-identified, if K rows of factor loadings, $\lambda_{i_k\cdot}$, $k = 1, \ldots, K$, each assigned to a different set, $\mathcal{I}_{w_{i_k}} \neq \mathcal{I}_{w_{i_l}}$, $k \neq l$, are full rank set identified and include at least $K(K-1)/2$ zero loading restrictions.*

Remark 3.1. The minimum number of K non-empty sets is given by the fact that factor identification is based on K populated and distinct rows of the loading matrix. Both conditions hold for all identification restrictions usually pre-imposed on loading matrices (see Subsection 2.3). For example, the conditions do not hold for Λ in (4), they hold for $\tilde{\Lambda}$ and Λ^{GLT}.

Remark 3.2. Set identification is not satisfied if there are two or more columns in Λ with the same structure of non-zero loadings. In this case, applying a QR decomposition to these columns reveals whether they have full rank. If they do, the QR decomposition induces a GLT structure with additional zero constraints, while preserving the loading structure in the remaining columns.

Remark 3.3. Mode identification is also obtained upon re-ordering appropriately the K elements chosen to evaluate full rank set identification as leading units of the data set. Λ may then be rotated into a GLT structure, inducing at least $K(K-1)/2$ zero restrictions required for mode identification.

Remark 3.4. Generally, more than one mode-identified sparse solution may underlie empirical data. In these cases, the mode of interest will be determined by e.g. the sparsest representation or the factors and their interpretation of interest. In this sense, set-based mode identification encompasses multiple mode-identified solutions (Kaufmann and Pape, 2023).

The matrix Λ in the example given by Jennrich (1978), see also Footnote 5, can be used to illustrate Remarks 3.3 and 3.4. Both matrices, the original matrix Λ and the Householder-transformed matrix ΛC, can be re-ordered by suitable permutation matrices P_{N_1} and P_{N_2},

respectively, to obtain loading matrices mode-identified by GLT and PLT, respectively:

$$P_{N_1}\Lambda = \begin{pmatrix} 1.03 & 0 & 0 & 0 \\ 1.42 & -0.79 & 0 & 0 \\ 1.41 & 0 & 1.12 & 0 \\ 1.63 & -0.81 & 0.31 & 0 \\ 0.67 & 0.89 & -0.10 & -1.09 \\ -1.21 & 0 & 0 & 0.03 \\ 0.72 & -1.07 & 0.32 & -1.21 \\ 0.81 & -0.34 & 0.23 & -0.41 \\ 0.73 & 0.33 & -0.16 & -0.77 \\ -0.30 & -0.75 & 0.63 & 0.37 \end{pmatrix}, \; P_{N_2}\Lambda C = \begin{pmatrix} 0.99 & 0 & 0 & 0 \\ 1.42 & -0.79 & 0 & 0 \\ 1.41 & 0 & 1.12 & 0 \\ 0.67 & 0.89 & -0.10 & -1.09 \\ -0.99 & 0.38 & -0.27 & 0.50 \\ 1.52 & 0.37 & -0.69 & 0.55 \\ 1.67 & -0.73 & 0.26 & 0.09 \\ 0.84 & -0.35 & 0.24 & -0.43 \\ 0.76 & 0.38 & -0.21 & -0.70 \\ 0 & -0.20 & 0.24 & 1.05 \end{pmatrix}.$$

3.2 Global identification: Sato's $\mathcal{O}(K^2)$ algorithm

Full rank set identification of Λ does not guarantee global identification (see Subsection 2.2), and we may refer to Sato (1992)'s counting rule to evaluate whether this necessary condition is fulfilled. Based on the sets $|\mathcal{I}_w| > 0$, we may re-frame the counting rule as a set-based condition.

Condition 3.3. *Sato's set-based counting rule: A necessary condition for a mode-identified (sparse) factor loading matrix to satisfy the condition of Theorem 5.1 in AR56 is that for all non-empty sets $\mathcal{I}_w$, the following condition holds:*

$$\sum_{x=1}^{2^K-1} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_w)\neq\varnothing\}} \geq 2|\mathcal{I}_w| + 1, \; \textit{for all } \mathcal{I}_w \textit{ with } |\mathcal{I}_w| > 0,$$

where N_x is the number of elements assigned to $\mathcal{I}_x$.

With an increasing number of factors, a full evaluation of Condition 3.3 becomes computationally expensive, as the number of sets to evaluate, $2^K - 1$, grows exponentially in K. In the following, we propose two algorithms to evaluate Condition 3.3, which both need a considerably lower number of evaluations. The outcome of the algorithms, although not sufficient, is necessary for Condition 3.3 to hold. Nevertheless, as demonstrated by simulation, both algorithms are highly reliable.

The first algorithm, Sato's $\mathcal{O}(K^2)$ algorithm, needs a number of evaluations that grows at most by K^2 rather than 2^K.

Algorithm 1. *Sato's $\mathcal{O}(K^2)$ algorithm*

1. Set $k = K$, the number of columns in Λ.
2. Delete zero rows from Λ, and determine the number of rows of Λ, which is n.
   If $n < 2k + 1$, the counting rule is violated, and the algorithm returns an error (stops).
3. For all $j = 1, \ldots, k$,
   a) count the number of nonzero elements in the j^{th} column of Λ, and denote it as $c_j = \sum_{w|I_w \supseteq \{j\}} N_w$. If any $c_j < 3$, the counting rule is violated, and the algorithm returns an error (stops).
   b) same as a), and determine $N_{\mathcal{I}_{1,j}}$.
4. a) Determine $\mathcal{J} = \{j^*|c_{j^*} = \max_j(c_j)\}$.
   b) Determine $\mathcal{J} = \{j^*|N_{\mathcal{I}_{1,j^*}} = \max_j(N_{\mathcal{I}_{1,j}})\}$.

   If $|\mathcal{J}| = 1$, choose $j^* \in \mathcal{J}$. Otherwise, if $|\mathcal{J}| > 1$, choose $j^* \in \mathcal{J}$ at random.
   If $|\mathcal{J}| = 0$, choose $j^* \in \{1, \ldots, k\}$ at random.
   Delete column j^* from Λ.
5. Set $k := k - 1$.
   If $k \geq 1$, proceed with step 2.
   If $k = 0$, Λ most probably satisfies the counting rule.

Remark 3.5. The algorithm includes two variants, a) and b) in Steps 3. and 4.. In variant a) the column with a maximum number of non-zero loadings is eliminated, while in variant b) the column with the maximum number of exclusive loadings is discarded.

The second algorithm relies in the first place on the evaluation of populated non-empty sets, $\mathcal{I}_w$ with $N_w > 0$, and included subsets, $\mathcal{I}_v \subseteq \mathcal{I}_w$. In particular for matrices of increasing factor dimension with a large degree of sparsity, the number of populated non-empty sets is potentially much lower than $2^K - 1$. Focusing on these may considerably accelerate the evaluation of the counting rule. We formulate the following condition:

Condition 3.4. *Sato's reduced set-based counting rule: A necessary condition for a mode-identified (sparse) factor loading matrix to satisfy the condition of Theorem 5.1 in AR56 is that for all populated non-empty sets $\mathcal{I}_w$ with $N_w > 0$, the following condition holds:*

$$\sum_{x|N_x>0} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_v) \neq \varnothing\}} \geq 2|\mathcal{I}_v| + 1, \text{ for all } \mathcal{I}_v \subseteq \mathcal{I}_w \text{ with } |\mathcal{I}_w| > 0 \text{ and } N_w > 0,$$

where N_x is the number of elements assigned to $\mathcal{I}_x$.

Remark 3.6. Condition 3.4 is equivalent to Condition 3.3 if the set $\{1, \ldots, K\}$ is populated.

The second algorithm uses Condition 3.4.

Algorithm 2. *Sato's set-based algorithm:*

1. Set $k = K$, the number of columns in Λ.

2. Delete zero rows from Λ, and determine the number of rows of Λ, which is n.
   If $n < 2k + 1$, the counting rule is violated, and the algorithm returns an error (stops).

3. Determine the populated non-empty sets, $\mathcal{I}_w$ with $N_w > 0$ and evaluate the reduced set-based Condition 3.4.
   If Condition 3.4 does not hold, the counting rule is violated, and the algorithm returns an error (stops).
   If Condition 3.4 holds and $N_{2^k-1} > 0$, Λ most probably satisfies the counting rule and the algorithm stops.

4. Determine $\mathcal{J} = \{j | j \in \mathcal{I}_{1,j}, N_w > 0\}$. If $|\mathcal{J}| = 0$, set $\mathcal{J} = \{j | j = 1, \ldots, K\}$.
   Determine $\mathcal{J}^* = \left\{ j^* | c_{j^*} = \max_j(c_j), c_j = \sum_{w | I_w \supseteq \{j\}} N_w, j \in \mathcal{J} \right\}$.
   If $|\mathcal{J}^*| = 1$, choose $j^* \in \mathcal{J}$, the column with the maximum number of non-zero loadings, and delete column j^* from Λ.
   If $|\mathcal{J}^*| > 1$, choose (Factor) j^+ with the least number of cross-loaded units, i.e. units loaded by other factors, $\mathcal{J}^+ = \left\{ j^+ | j^+ = \min_{j^*} \left(\sum_{w | \mathcal{I}_w \supset \{j^*\}} N_w \right), j^* \in \mathcal{J}^* \right\}$. If $|\mathcal{J}^+| > 1$ choose j^+ randomly. Delete column j^+ from Λ.

5. Set $k := k - 1$.
   If $k \geq 1$, proceed with step 2.
   If $k = 0$, Λ most probably satisfies the counting rule.

Remark 3.7. In step 3, N_{2^k-1} corresponds to the number of rows populating the set $\mathcal{I}_{2^k-1} = \{1, \ldots, k\}$. If the set is populated, evaluations correspond to evaluating Condition 3.3, see Remark 3.6. If $k = K$, Λ satisfies Condition 3.3.

4 Efficiency and illustration

In this section, we evaluate the efficiency of the proposed algorithms by simulation and present an empirical application to illustrate the set-based identification procedure. Run-time efficiency is of particular interest in applications which rely on Bayesian simulation methods to obtain posterior inference.

4.1 Efficiency

Table 1 documents the run-time efficiency of Sato's $\mathcal{O}(K^2)$ algorithm in both variants (see Remark 3.5) and Sato's set-based algorithm, comparing them to the set evaluation

Table 1: Runtimes (in seconds) for evaluating 10,000 matrices of dimension $5K \times K$, with different shares of matrices satisfying the counting rule. Evaluation by Condition 3.3 (Cond.3.3.), Sato's $\mathcal{O}(K^2)$ algorithm, variant a) (A1a), and variant b) (A1b), and the set-based algorithm (A2).

	90%				50%				10%			
K	Cond.3.3	A1a	A1b	A2	Cond.3.3	A1a	A1b	A2	Cond.3.3	A1a	A1b	A2
3	0.19	0.23	0.34	1.40	0.09	0.19	0.24	1.12	0.09	0.13	0.14	0.77
4	0.32	0.30	0.45	2.52	0.17	0.23	0.30	2.02	0.10	0.14	0.17	1.21
5	0.66	0.34	0.49	3.95	0.31	0.26	0.33	2.92	0.15	0.15	0.19	1.67
6	1.68	0.40	0.59	6.00	0.67	0.30	0.39	4.24	0.22	0.16	0.20	2.36
7	3.13	0.46	0.69	8.38	1.36	0.33	0.45	5.76	0.42	0.18	0.23	3.25
8	5.91	0.53	0.81	11.46	2.85	0.38	0.52	7.85	0.75	0.20	0.24	4.56
9	11.67	0.61	0.90	15.88	6.63	0.43	0.57	10.99	1.52	0.22	0.26	6.28
10	22.08	0.69	1.03	21.57	13.75	0.51	0.65	15.18	2.99	0.23	0.29	8.65
11	48.04	0.76	1.16	29.04	26.12	0.53	0.73	20.90	5.96	0.25	0.30	12.03
12	97.00	0.89	1.30	40.11	54.42	0.60	0.83	29.85	12.05	0.28	0.34	16.48
13	200.36	1.11	1.44	54.00	111.81	0.65	0.91	40.85	26.46	0.29	0.35	22.64
14	418.52	1.22	1.61	73.43	236.18	0.69	1.02	52.95	55.49	0.31	0.42	31.10
15	868.94	1.35	1.82	97.60	482.05	0.76	1.13	69.28	122.25	0.33	0.43	42.43
16	1777.34	1.29	2.00	131.54	997.24	0.86	1.23	94.23	225.63	0.35	0.47	58.09

as required by Condition 3.3. We generate 10,000 sparse matrices of dimensions $5K \times K$ for $K \in \{3, \ldots, 16\}$ for each of three scenarios, with 90%, 50% and 10% of the simulated matrices satisfying the counting rule, respectively. Note that for matrices satisfying the counting rule, Condition 3.3 requires evaluating all $2^K - 1$ sets, while for matrices not satisfying the counting rule, evaluation stops as soon as a set violates the counting rule. Hence, for scenarios including a high percentage of matrices satisfying the counting rule (high-percentage scenarios), runtimes will be longer than for scenarios with a low percentage of matrices satisfying the counting rule (low-percentage scenarios). Both algorithms likewise require more time for high-percentage scenarios. Table 1 shows that while for very small K, evaluation based on Condition 3.3 (Cond.3.3) is faster than using Sato's $\mathcal{O}(K^2)$ algorithm (A1a and A1b), the latter is much faster for small- to medium-sized models. Runtimes are lower by a factor of 1,000 for the largest dimension. Sato's set-based algorithm (A2) is faster than the counting rule in larger settings, around $K \geq 10$ in the 90% and 50% scenarios, and for $K \geq 13$ in the 10% scenario. Runtimes are lower by a factor of around 4 to 10 for the largest dimensions.

In Table 2, we extend the simulation to seven scenarios of percentages of matrices satisfying the counting rule, simulating again 10,000 matrices of dimension $5K \times K$ with $K \in \{3, \ldots, 16\}$ for each scenario. We evaluate whether the algorithms correctly identify matrices as satisfying the counting rule or not. Note that the only error possible is a false positive, i.e., the algorithm fails to identify a matrix not satisfying the counting rule. This is due to the fact that Sato's $\mathcal{O}(K^2)$ algorithm only evaluates a minimal number, whereas Sato's set-based algorithm evaluates a lower number of sets than required by Sato's rule (Condition 3.3).

Table 2: Number of false positive results for 10,000 simulated matrices of dimension $5K \times K$ for various percentages of matrices satisfying the counting rule, as detected by variant a) of the $\mathcal{O}(K^2)$ algorithm.

K	3	4	5	6	7	8	9	10	11	12	13	14	15	16
share														
95%	0	0	0	0	0	0	0	0	0	0	1	0	0	1
90%	0	0	0	0	0	0	0	0	0	0	0	0	0	0
75%	0	0	0	0	0	0	0	0	0	0	0	0	0	1
50%	1	1	0	0	0	0	0	0	0	0	1	0	0	1
25%	1	1	1	0	0	0	1	0	0	0	1	0	0	1
10%	0	1	1	0	0	0	1	0	0	0	1	0	0	1
0%	1	1	1	0	0	0	1	0	0	0	1	0	0	1

Both variant b) of Sato's $\mathcal{O}(K^2)$ and Sato's set-based algorithms return no false positives. Variant a) of Sato's $\mathcal{O}(K^2)$ algorithm, on the other hand, returns very few false positives. For example, in the 95% scenario there could be up to 500 false positives, whereas the algorithm returns at most one for each K. The same holds for the lower-percentage scenarios, where the number of false positives could be even higher.

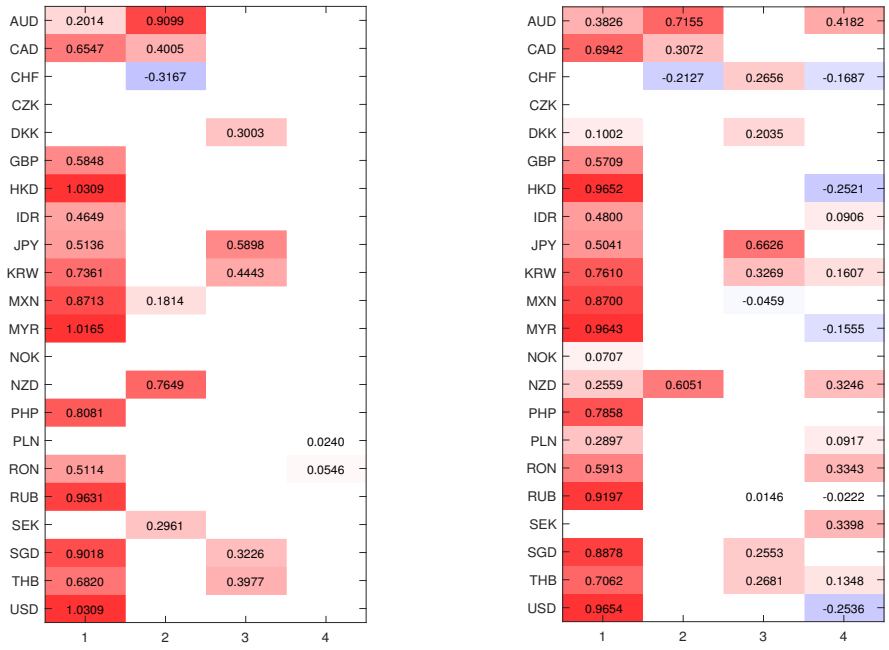
The matrices were simulated under the assumption of linearly increasing sparsity in factor order, which reflects usual properties of empirical estimates. Similar results were found for exponentially increasing sparsity. For - rather unrealistic - scenarios with uniformly very sparse patterns across factors, we observe more false positives for both algorithms. The number is still very low even in these cases, however.

We conclude that in empirical applications, a larger share of draws violating the counting rule indicates that the model is misspecified, for example overfitting the data.

4.2 Illustration

For the empirical illustration, we fit Model (1) to monthly log returns in exchange rates for 22 currencies against the euro, covering the period from January 2000 to December 2007, see also Frühwirth-Schnatter et al. (2024). It seems odd to fit a static factor model to time series usually displaying some (common) persistence. However, static independent factors correspond to the unconditional representation of independent, scaled conditional or dynamic factors.[13] Admittedly, extracted static factors may be more difficult to interpret in a time series setting. However, we are not primarily interested in factor interpretation. Rather, our objective is to illustrate the evaluation of factor identification based on the proposed set identification conditions. Moreover, our estimates are directly comparable to those presented in Frühwirth-Schnatter et al. (2024), who estimate a static factor model

[13] A dynamic, conditional independent process may underlie each static factor k, $k = 1, \ldots, K$, in Model (1): $f_{kt} = \phi_k f_{k,t-1} + \nu_{kt}$, $\nu_{kt} \sim N(0, \frac{1}{1-\phi_k^2})$ corresponds to the unconditional or static representation $f_{kt} \sim N(0, 1)$. The persistence ϕ_k remains unidentified in Model (1), however.



(a) Unconstrained (b) Sparse

Figure 2: $K = 4$. Panel (a) Posterior mean of CFA estimated non-zero factor loadings. Panel (b) Posterior median of post-processed sparse MCMC output. All draws fulfill the counting rule.

imposing a GLT structure on the factor loading matrix.

We present and evaluate the output of two estimates. Both estimates are based on the posterior output of Gibbs samplers. The first estimate is based on the output obtained from an unconstrained rotation sampler, under a normal prior for the factor loadings, (Aßmann et al., 2016). The Markov chain Monte Carlo (MCMC) output is then post-processed by rotation, to optimize on a sparse representation of the factor loading matrix. The output presented below is based on a confirmatory factor model estimate, which reflects the sparse representation obtained from the post-processed posterior output. The second estimate is based on the post-processed output of a permutation sampler, under a hierarchical spike-and slab prior inducing sparsity into the factor loading matrix (West, 2003; Kaufmann and Schumacher, 2019). Both samplers and the post-processing procedures are detailed in Kaufmann and Pape (2023).

We first analyze a model with $K = 4$ factors. Figure 2 displays heat plots of non-zero factor loadings identified from the posterior output of each approach. Posterior inference based on the output of the unconstrained rotation sampler (Panel (a)) identifies a sparser factor loading matrix than posterior inference based on the sparse permutation sampler (Panel (b)). Table 3 lists the set population, that is the number of rows in Λ assigned to each of the subsets $\mathcal{I}_w \subseteq \{1, 2, 3, 4\}$. Both factor loading matrices displayed in Figure 2 are set-identified, as the number of populated non-empty subsets for each (8 and 9, respectively) are larger than $K = 4$, and these sets span the factor space $\bigcup_{w|N_w>0} \mathcal{I}_w = \{1, \ldots, 4\}$.

Table 3: Set population. Model estimates for $K = 4$ and $K = 3$, non-zero loadings identified based on the post-processed MCMC output of the unconstrained rotation and the sparse permutation sampler.

Set	Elements	$N_{w,K=4}$		$N_{w,K=3}$	
		Unconstrained	Sparse	Unconstrained	Sparse
$\mathcal{I}_0 = \mathcal{I}_{0,1}$	$= \{\}$	2	1	2	2
$\mathcal{I}_1 = \mathcal{I}_{1,1}$	$= \{1\}$	7	3	9	4
$\mathcal{I}_2 = \mathcal{I}_{1,2}$	$= \{2\}$	3	0	3	1
$\mathcal{I}_3 = \mathcal{I}_{2,1}$	$= \{1,2\}$	3	2	3	8
$\mathcal{I}_4 = \mathcal{I}_{1,3}$	$= \{3\}$	1	0	1	1
$\mathcal{I}_5 = \mathcal{I}_{2,2}$	$= \{1,3\}$	4	4	4	4
$\mathcal{I}_6 = \mathcal{I}_{2,3}$	$= \{2,3\}$	0	0	0	1
$\mathcal{I}_7 = \mathcal{I}_{3,1}$	$= \{1,2,3\}$	0	0	0	1
$\mathcal{I}_8 = \mathcal{I}_{1,4}$	$= \{4\}$	1	1		
$\mathcal{I}_9 = \mathcal{I}_{2,4}$	$= \{1,4\}$	1	6		
$\mathcal{I}_{10} = \mathcal{I}_{2,5}$	$= \{2,4\}$	0	0		
$\mathcal{I}_{11} = \mathcal{I}_{3,2}$	$= \{1,2,4\}$	0	2		
$\mathcal{I}_{12} = \mathcal{I}_{2,6}$	$= \{3,4\}$	0	0		
$\mathcal{I}_{13} = \mathcal{I}_{3,3}$	$= \{1,3,4\}$	0	3		
$\mathcal{I}_{14} = \mathcal{I}_{3,4}$	$= \{2,3,4\}$	0	1		
$\mathcal{I}_{15} = \mathcal{I}_{4,1}$	$= \{1,2,3,4\}$	0	0		

When choosing the British pound (GBP), the New Zealand dollar (NZD), Danish kroner (DKK) and the Polish zloty (PLN), we obtain submatrices of the following form:

$$\Lambda_{a,\text{Panel (a)}} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \Lambda_{a,\text{Panel (b)}} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & * \\ * & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

These submatrices are of full-rank (see values in Figure 2) and contain at least $4(4-1)/2 = 6$ zero elements. While the factor loading matrix corresponding to $\Lambda_{a,\text{Panel (a)}}$ is PLT identified, PLT identification of the factor loading matrix corresponding to $\Lambda_{a,\text{Panel (b)}}$ may be induced by exchanging Rows 2 and 4 in $\Lambda_{a,\text{Panel (b)}}$ combined with exchanging Columns 2 and 4 of the whole factor loading matrix Λ. We conclude that both estimates are set-based mode-identified.

However, applying Sato's $\mathcal{O}(K^2)$ and the set-based algorithms to the factor loading matrix displayed in Panel (a), immediately reveals that the counting rule is violated. In Panel (a), we see that, although $n = 20 > 9 = 2K + 1$, Algorithm 1 determines $c_4 = 2 < 3$ in the very first iteration. Table 4 displays the result based on Algorithm 2, where $N_v = 2$ for $\mathcal{I}_v = \{4\}$ violates Sato's reduced set-based counting rule (Condition 3.4) also in the first iteration. On the other hand, according to both algorithms the factor loading matrix in Panel (b) of Figure 2 does not violate the counting rule, see the results based on Algorithm 2 tabulated in Appendix A. Table A.1 shows that the algorithm stops at the

Table 4: $K = 4$. Set-based evaluation of the factor loading matrix in Panel (a) of Figure 2. The columns $\mathcal{I}_v = \mathcal{I}_w$ evaluate the number of rows of non-zero factor loadings for column sets $\mathcal{I}_v$ corresponding to populated sets $|\mathcal{I}_w| > 0,\ N_w > 0$.

	$\mathcal{I}_v = \mathcal{I}_w$						
$\mathcal{I}_w$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{4\}$	$\{1,4\}$
$\{1\}$	7		7		7		7
$\{2\}$		3	3				
$\{1,2\}$	3	3	3		3		3
$\{3\}$				1	1		
$\{1,3\}$	4		4	4	4		4
$\{4\}$						1	1
$\{1,4\}$	1		1		1	1	1
$\sum_{x\|N_x>0} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_v)\neq\varnothing\}}$	15	6	18	5	16	2	16
$2\|\mathcal{I}_v\| + 1$	3	3	5	3	5	3	5
	✓	✓	✓	✓	✓	×	✓

second iteration. In both iterations, the column sets $\mathcal{I}_v$ to evaluate have to be completed by those subsets $\mathcal{I}_v \subset \mathcal{I}_w$ included in the non-empty, populated sets $\mathcal{I}_w$. The number of sets evaluated in the first iteration (13) is lower than the number required for full evaluation (15) according to Condition 3.3. The number of evaluated sets in the second iteration corresponds to an evaluation of Condition 3.3 for $k = 3$. The algorithm stops and returns that Λ most probably satisfies the counting rule. Nevertheless, in Figure 2, Panel (b), we also see that 7 out of 13 factor loadings of Factor 4 are smaller than 0.20 (in absolute terms), and it turns out that the posterior means of Factors 4 and 2 correlate with 0.35, which is quite substantial. This suggests that a model estimated with $K = 3$ may improve the consistency across both posterior inferences.

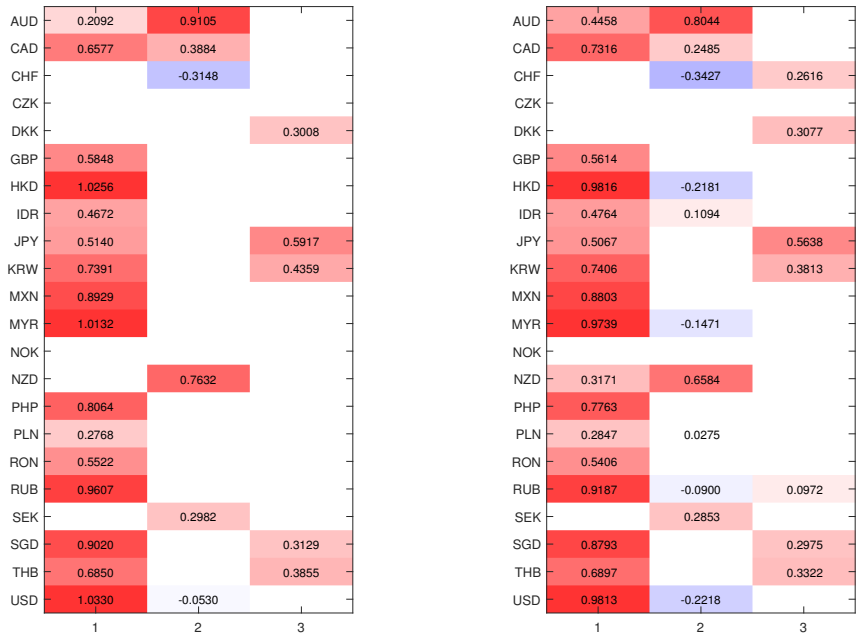
Figure 3 displays the heatplots of non-zero factor loadings for a model estimated for $K = 3$. If we disregard the small (in absolute terms) factor loadings in Panel (b), we observe that results and hence factor interpretation are quite consistent across posterior outputs. The set population is displayed in the last two columns of Table 3. Both outputs are set-identified, as more that $K = 3$ sets are populated and the union over populated, non-empty sets spans the factor space, $\{1, 2, 3\}$.

Choosing the British pound (GBP), the New Zealand dollar (NZD) and the Danish kroner (DKK), we obtain submatrices

$$\Lambda_{a,\text{Panel (a)}} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \qquad \text{and} \qquad \Lambda_{a,\text{Panel (b)}} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

which both are of full rank (see values in Figure 3) and contain more than 3 zero elements. Both posterior outputs are set-based mode-identified.

Sato's $\mathcal{O}(K^2)$ algorithm now goes through three iterations for both outputs. Figure 4



(a) Unconstrained (b) Sparse

Figure 3: $K = 3$. Panel (a) Posterior mean of CFA estimated non-zero factor loadings. Panel (b) Posterior median of post-processed sparse MCMC output. All draws fulfill the counting rule.

illustrates the steps for the matrix of Panel (a) in Figure 3. Factor 1, with 16 non-zero factor loadings is eliminated first. Of the remaining two factors, Factor 1 with 6 non-zero loadings is eliminated. The last remaining factor is identified with 5 non-zero factor loadings. Algorithm 1 would remove factors in the same order for the loading matrix displayed in Panel (b). The set-based algorithm removes the factors in the same order when applied to the matrix displayed in Panel (a). For the matrix displayed in Panel (b), the algorithm confirms that Condition 3.4 is fulfilled and stops in the first iteration (see Table 5), given that the number of evaluations corresponds to an evaluation of Condition 3.3 for $K = 3$.

Overall, the evaluation of the posterior outputs suggests that a model with $K = 3$ is appropriate for fitting Model 1 to these exchange rate data. Post-processing the output of the unconstrained sampler by rotation to optimize towards a sparse factor loading matrix yields a sparser posterior output in general. Nevertheless, the posterior factor loading matrices and hence, the interpretation of factors, turns out to be quite consistent across the posterior output of both approaches.

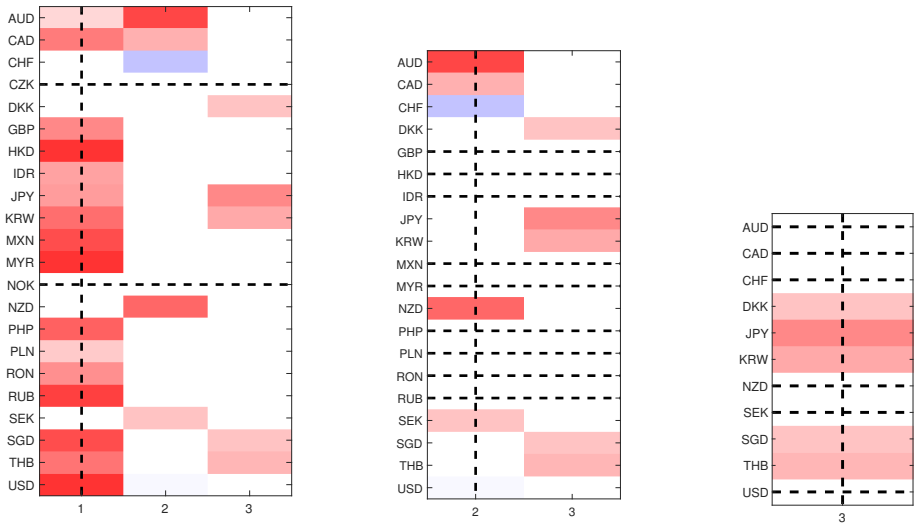


Figure 4: $K = 3$, applying Algorithm 1 to the loading matrix displayed in Panel (a) of Figure 3. Loading matrices after elimination of Factor 1 (middle panel) and Factor 2 (right panel).

Table 5: $K = 3$, set-based evaluation of the factor loading matrix in Panel (b) of Figure 2. The columns $\mathcal{I}_v = \mathcal{I}_w$ evaluate the number of rows of non-zero factor loadings for column sets $\mathcal{I}_v$ corresponding to populated sets $|\mathcal{I}_w| > 0,\ N_w > 0$.

	$\mathcal{I}_v = \mathcal{I}_w$						
$\mathcal{I}_w$	$\{1\}^*$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	4		4		4		4
$\{2\}$		1	1		1	1	1
$\{1,2\}$	8	8	8		8	8	8
$\{3\}$				1	1	1	1
$\{1,3\}$	4		4	4	4	4	4
$\{2,3\}$		1	1	1	1	1	1
$\{1,2,3\}$	1	1	1	1	1	1	1
$\sum_{x\|N_x>0} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_v)\neq\varnothing\}}$	17	11	19	7	20	16	20
$2\|\mathcal{I}_v\| + 1$	3	3	5	3	5	5	7
	✓	✓	✓	✓	✓	✓	✓

5 Conclusion

Following the geometric representation of factor models by Lawley and Maxwell (1971), we present a geometric approach to identification of factor models. After reviewing some published results relevant for and related to our discussion, we introduce the concept of set identification. We formulate conditions to evaluate whether a (posterior) estimate of a factor model is globally and set-based mode- or rotation-identified. In particular, the set conditions are order-invariant, i.e. mode or rotational identification does not rely on specific restrictions imposed on the leading $K \times K$ submatrix of the factor loading matrix. Set identification is not sufficient for global identification, though. Therefore, we make use of the counting rule proposed by Sato (1992), which we re-formulate in terms of set conditions. We propose two efficient algorithms to assess global identification based on the counting rule. Compared with the full evaluation of Sato's counting rule, both algorithms reduce the number and the time of evaluations considerably, in particular for long MCMC chains of large factor models.

A comprehensive simulation study reveals that variant b) of Algorithm 1 and Algorithm 2 always correctly identify matrices violating the counting rule, whereas variant a) of Algorithm 1 performs correctly in almost all cases. Compared with a full evaluation of the counting rule, runtimes improve by a factor of 1,000 for Algorithm 1, and up to a factor of 10 for Algorithm 2 in large ($K > 10$) dimensions. To illustrate the procedures, we estimate a static factor model for exchange rate data, using two approaches to induce sparsity into the factor loading matrix. While set-based mode identification is confirmed across both posterior outputs for different settings ($K = \{4, 3\}$), global identification is confirmed for both posterior outputs in the setting $K = 3$ only.

We conclude that a set-based approach to factor identification and the algorithms proposed in the paper are very useful. They provide an order-invariant procedure to evaluate identification properties of (posterior) estimates of factor models.

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A Additional results

Table A.1: $K = 4$, Set-based evaluation of the factor loading matrix in Panel (b) of Figure 2. The factor eliminated is indicated by a *. Columns $\mathcal{I}_v = \mathcal{I}_w$ evaluate the number of rows with non-zero loadings of the column sets $\mathcal{I}_v$ corresponding to populated sets $\mathcal{I}_w$, $N_w > 0$, columns $\mathcal{I}_v \subset \mathcal{I}_w$ evaluate the number of rows with non-zero loadings of additional column subsets included in populated sets.

	Iteration 1												
	$\mathcal{I}_v = \mathcal{I}_w$								$\mathcal{I}_v \subset \mathcal{I}_w$				
$\mathcal{I}_w$	$\{1\}^*$	$\{1,2\}$	$\{1,3\}$	$\{4\}$	$\{1,4\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$\{1\}$	3	3	3		3	3	3						
$\{1,2\}$	2	2	2		2	2	2	2	2		2	2	
$\{1,3\}$	4	4	4		4	4	4	4		4	4		4
$\{4\}$				1	1	1	1	1				1	1
$\{1,4\}$	6	6	6	6	6	6	6	6				6	6
$\{1,2,4\}$	2	2	2	2	2	2	2	2	2		2	2	2
$\{1,3,4\}$	3	3	3	3	3	3	3	3		3	3	3	3
$\{2,3,4\}$		1	1	1	1	1	1	1	1	1	1	1	1
$\sum_{x\|N_x>0} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_v)\neq\varnothing\}}$	20	21	21	13	22	22	22	19	5	8	12	15	17
$2\|\mathcal{I}_v\|+1$	3	5	5	3	5	7	7	7	3	3	5	5	5
	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

	Iteration 2						
	$\mathcal{I}_v = \mathcal{I}_w$						$\mathcal{I}_v \subset \mathcal{I}_w$
$\mathcal{I}_w$	$\{1\}$	$\{2\}$	$\{3\}^*$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{1,2\}$
$\{1\}$	1			1		1	1
$\{2\}$		4			4	4	4
$\{3\}$			7	7	7	7	
$\{1,3\}$	2		2	2	2	2	2
$\{2,3\}$		3	3	3	3	3	3
$\{1,2,3\}$	1	1	1	1	1	1	1
$\sum_{x\|N_x>0} N_x 1_{\{(\mathcal{I}_x \bigcap \mathcal{I}_v)\neq\varnothing\}}$	4	8	13	14	17	18	11
$2\|\mathcal{I}_v\|+1$	3	3	3	5	5	7	5
	✓	✓	✓	✓	✓	✓	✓