Lecture 5A: Leland-type Models

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Leland Models

  - If you want to combine model with data, this is the typical setting
- A dynamic version of traditional trade-off model, but capital structure decision is static
  - Trade-off model: a firm’s leverage decision trades off the tax benefit with bankruptcy cost
- Relative to the previous literature (say Merton’s 1974 model), Leland setting emphasizes equity holders can decide default timing ex post
  - So-called "endogenous default," an useful building block for more complicated models
  - Merton 1974 setting: given $V_T$ distribution, default if $\tilde{V}_T < F_T$. No default before $T$ and the path of $V_t$ does not matter
Firm and Its Cash Flows

- A firm’s asset-in-place generates cash flows at a rate of $\delta_t$
  - Over interval $[t, t + dt]$ cash flows is $\delta_t dt$
  - Leland ’94, state variable unlevered asset value $V_t = \frac{\delta_t}{r - \mu}$ (just relabeling)
- Cash flow rate follows a Geometric Brownian Motion (with drift $\mu$ and volatility $\sigma$)
  \[
  \frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t
  \]
  - $\{Z_t\}$ is a standard Brownian motion (Wiener process):
    - $Z_t \sim \mathcal{N}(0, t)$, $Z_t - Z_s$ is independent of $\mathcal{F}(\{Z_u < s\})$
  - Given $\delta_0$, $\delta_t = \delta_0 \exp\left((\mu - 0.5\sigma^2) t + \sigma Z_t\right) > 0$
  - Arithmetic Brownian Motion: $d\delta_t = \mu dt + \sigma dZ_t$ so
    - $\delta_t = \delta_0 + \mu t + \sigma Z_t$
- Persistent shocks, i.i.d. return. Today’s shock $dZ_t$ affects future level of $\delta_s$ for $s > t$
- One interpretation: firm produces one unit of good per unit of time, with market price fluctuating according to a GBM
- In this model, everything is observable, i.e. no private information
Debt as Perpetual Coupon

- Firm is servicing its debt holders by paying coupon at the rate of $C$
  - Debt holders are receiving cash flows $Cdt$ over time interval $[t, t + dt]$
- Debt tax shield, with tax rate $\tau$
- Debt is deducted before calculating taxable income implies that debt can create DTS
- The previous cash flows are after-tax cash flows, so before-tax cash flows are $\frac{\delta_t}{1 - \tau}$
  - So-called Earnings Before Interest and Taxes (EBIT)
- By paying coupon $C$, taxable earning is $\frac{\delta_t}{1 - \tau} - C$, so equity holders’ cash flows are
  $$\left(\frac{\delta_t}{1 - \tau} - C\right) (1 - \tau) = \delta_t - (1 - \tau) C$$
- The firm investors in total get (Modigliani-Miller idea)
  $$\frac{\delta_t}{1 - \tau} - C + C = \frac{\delta_t}{\text{Firm's Asset}} + \tau C$$
Endogenous Default Boundary

- Equity holders receiving $\delta_t$ which might become really low, but is paying constant $(1 - \tau) C$
- When $\delta_t \to 0$, holding the firm almost has zero value—then why pay those debt holders?
- Equity holders default at $\delta_B > 0$ where equity value at $\delta_B$ has $E(\delta_B) = 0$ and $E'(\delta_B) = 0$
  - Value matching $E(\delta_B) = 0$, just says that at default equity holders recover nothing
  - Smooth pasting $E'(\delta_B) = 0$, optimality: equity can decide to wait and default at $\delta_B - \epsilon$, but no benefit of doing so
- At bankruptcy, some deadweight cost, debt holders recover a fraction $1 - \alpha$ of first-best firm value $(1 - \alpha) \delta_B / (r - \mu)$
  - First-best unlevered firm value $\delta_B / (r - \mu)$, Gordon growth formula
- Two steps:
  1. Derive debt $D(\delta)$ and equity $E(\delta)$, given default boundary $\delta_B$
  2. Using smooth pasting condition to solve for $\delta_B$
General Solution for GBM process with Linear Flow Payoffs

\[ rV(y) = f(y) + V'(y)\mu(y) + \frac{1}{2}\sigma^2(y) V''(y) \]

- **required return**
- **flow (dividend) payoff**
- **local change of value function**

- Leland setting, \( f(y) = a + by \), \( \mu(y) = \mu y \), and \( \sigma(y) = \sigma y \)

- It is well known that the general solution to \( V(y) \) is

\[ V(y) = \frac{a}{r} + \frac{b}{r - \mu} y + K_\gamma y^{-\gamma} + K_\eta y^\eta \]

where the "power" parameters are given by

\[ -\gamma = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} < 0, \]

\[ \eta = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} > 1 \]

- The constants \( K_\gamma \) and \( K_\eta \) are determined by boundary conditions
Debt Valuation (1)

- For debt, flow payoff is $C$ so

\[ D(\delta) = \frac{C}{r} + K\gamma \delta^{-\gamma} + K\eta \delta^\eta \]

- Two boundary conditions
  - When $\delta = \infty$, default never occurs, so $D(\delta = \infty) = \frac{C}{r}$ perpetuity. Hence $K\eta = 0$ (otherwise, $D$ goes to infinity)
  - When $\delta = \delta_B$, debt value is $\frac{(1-\alpha)\delta_B}{r-\mu}$. $D(\delta_B) = \frac{(1-\alpha)\delta_B}{r-\mu}$ implies that

\[
\frac{C}{r} + K\gamma \delta_B^{-\gamma} = \frac{(1-\alpha)\delta_B}{r-\mu} \quad \Rightarrow \quad K\gamma = \frac{\frac{(1-\alpha)\delta_B}{r-\mu} - \frac{C}{r}}{\delta_B^{-\gamma}}
\]
Debt Valuation (2)

- We obtain the closed-form solution for debt value

$$D(\delta) = \frac{C}{r} + \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \left( \frac{(1-\alpha)\delta_B}{r-\mu} - \frac{C}{r} \right)$$

$$= \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \frac{(1-\alpha)\delta_B}{r-\mu} + \left( 1 - \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \right) \frac{C}{r}$$

- Present value of 1 dollar contingent on default:

$$\mathbb{E} [e^{-r\tau_B}] = \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \text{ where } \tau_B = \inf \{ t : \delta_t < \delta_B \}$$

- The debt value can also be written in the following intuitive form

$$D(\delta) = \mathbb{E} \left[ \int_0^{\tau_B} e^{-rs} Cds + e^{-r\tau_B} \frac{(1-\alpha)\delta_B}{r-\mu} \right]$$

$$= \mathbb{E} \left[ \frac{C}{r} \left( - \int_0^{\tau_B} de^{-rs} \right) + e^{-r\tau_B} \frac{(1-\alpha)\delta_B}{r-\mu} \right]$$

$$= \mathbb{E} \left[ \frac{C}{r} (1 - e^{-r\tau_B}) + e^{-r\tau_B} \frac{(1-\alpha)\delta_B}{r-\mu} \right]$$
Equity Valuation (1)

- For equity, flow payoff is $\delta_t - (1 - \tau) C$, so
  $$E(\delta) = \frac{\delta}{r - \mu} - \frac{(1 - \tau) C}{r} + K_\gamma \delta^{-\gamma} + K_\eta \delta^\eta$$

- When $\delta = \infty$, equity value cannot grow faster than first-best firm value which is linear in $\delta$. So $K_\eta = 0$

- When $\delta = \delta_B$, we have
  $$E(\delta_B) = \frac{\delta_B}{r - \mu} - \frac{(1 - \tau) C}{r} + K_\gamma \delta_B^{-\gamma} = 0 \Rightarrow K_\gamma = \frac{(1 - \tau) C}{r} - \frac{\delta_B}{r - \mu}$$

Thus

$$E(\delta) = \frac{\delta}{r - \mu} - \frac{(1 - \tau) C}{r} + \left( \frac{(1 - \tau) C}{r} - \frac{\delta_B}{r - \mu} \right) \left( \frac{\delta}{\delta_B} \right)^{-\gamma}$$

Equity value if never defaults (pay $(1 - \tau) C$ forever)

Option value of default
Finally, smooth pasting condition

$$0 = \left. E'(\delta) \right|_{\delta=\delta_B}$$

$$= \frac{1}{r - \mu} + \left( \frac{(1 - \tau) C}{r} - \frac{\delta_B}{r - \mu} \right) (-\gamma) \left( \frac{\delta}{\delta_B} \right)^{-\gamma-1} \frac{1}{\delta_B} \bigg|_{\delta=\delta_B}$$

$$= \frac{1}{r - \mu} + (-\gamma) \left( \frac{(1 - \tau) C}{r\delta_B} - \frac{1}{r - \mu} \right)$$

Thus

$$\delta_B = (1 - \tau) C \frac{r - \mu}{r} \frac{\gamma}{1 + \gamma}$$
What if the firm can decide optimal coupon

- At \( t = 0 \), what is the optimal capital structure (leverage)?
- Given \( \delta_0 \) and \( C \), the total levered firm value \( v(\delta_0) = E(\delta_0) + D(\delta_0) \) is

\[
\frac{\delta_0}{r - \mu} + \frac{\tau C}{r} \left( 1 - \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \right) - \frac{\alpha \delta_B}{r - \mu} \left( \frac{\delta}{\delta_B} \right)^{-\gamma}
\]

Unlevered value \hspace{10cm} Tax shield \hspace{10cm} Bankruptcy cost

- Realizing that \( \delta_B \) is linear in \( C \), we can find the optimal \( C^* \) that maximizing the levered firm value to be

\[
C^* = \frac{\delta_0}{r - \mu} \frac{r (1 + \gamma)}{(1 - \tau) \gamma} \left( 1 + \gamma + \frac{\alpha \gamma (1 - \tau)}{\tau} \right)^{-1/\gamma}
\]

- Important observation: optimal \( C^* \) is linear in \( \delta_0 \)! So called scale-invariance
  - It implies that if the firm is reoptimizing, its decision is just some constant scaled by the firm size
Trade-off Theory: Economics behind Leland (1994)

- Benefit: borrowing gives debt tax shield (DTS)
- Equity holders makes default decision ex post
- The firm fundamental follows GBM, **persistent** income shocks
- After enough negative shocks, equity holders’ value of keeping the firm alive can be really low
- Debt obligation is fixed, so when $\delta_t$ is sufficiently low, it is optimal to default
  - Debt-overhang—Equity holders do not care if default impose losses on debt holders
- But, at time zero when equity holders issue debt, debt holders price default in $D(\delta_0)$
  - And equity holders will receive $D(\delta_0)$!
- Hence equity holders optimize $E(\delta_0) + D(\delta_0)$, realizing that coupon $C$ will affect DTS (positively) and bankruptcy cost (negatively)
- If equity holders can commit ex ante about ex post default behavior, what do they want to do?
There are two modifications relative to Leland (1994):

First, directly modelling pre-tax cashflows – so-called EBIT, rather than after-tax cashflows

It makes clear that there are three parties to share the cashflows: equity, debt, and government

When we take comparative statics w.r.t. tax rate $\tau$, in Leland (1994) you will ironically get that levered firm value $\uparrow$ when $\tau \uparrow$

- In Leland, raising $\tau$ does not change $\delta_t$ (which is after-tax cashflows)

In LGJ, after-tax cashflows are $(1 - \tau) \delta_t$, so raising $\tau$ lowers firm value
Second, more importantly, allowing for firms to upward adjust their leverage if it is optimal to do so in the future
- When future fundamental goes up, leverage goes down, optimal to raise more debt
- Need fix cost to do so—otherwise tend to do it too often

Key assumption for tractability: when adjusting leverage, the firm has to buy back all existing debt
- Say that this rule is written in debt covenants
- As a result, there is always one kind of debt at any point of time

After buying back, when equity holders decide how much debt to issue, they are solving the same problem again with new firm size
- But the model is scale invariant, so the solution is the same (except a larger scale)

$F$ face value. A firm with $(\delta, F)$ faces the same problem as $(k\delta, kF)$
Optimal Policies in LGJ

\[ \frac{\delta_B}{\delta_0} = \psi: \text{ default factor, } \frac{\delta_U}{\delta_0} = \gamma: \text{ leverage adjustment factor} \]

LGJ: can precommit to \( \gamma \). No precommitment in Fischer-Heinkel-Zechner (1989)
How Do We Model Finite Maturity

- Perpetual debt in Leland (1994). In practice debt has finite maturity.
- Debt maturity is very hard to model in a dynamic model.
- You can do exponentially decaying debt (Leland, 1994b, 1998).
- Rough idea: what if your debt randomly matures in a Poisson fashion with intensity $1/m$?
- Exponential distribution, the expected maturity is
  \[
  \int_{0}^{\infty} x \frac{1}{m} e^{-x/m} dx = m
  \]
- It is memoryless—if the debt has not expired, looking forward the debt price is always the same!
- Actually, you do not need random maturing. Exponential decaying coupon payment also works!
- So, debt value is $D(\delta)$, not $D(\delta, t)$ where $t$ is remaining maturity.
- If all debt maturity is i.i.d, large law of numbers say that at $[t, t + dt]$, $\frac{1}{m} dt$ fraction of debt mature.
Leland (1998)

- Using exponentially decaying finite maturity debt
- Equity holders can ex post choose risk

\[ \sigma \in \{ \sigma_H, \sigma_L \} \]

- Research question: how does asset substitution work in this dynamic framework? How does it depend on debt leverage and debt maturity?
- Typically with default option, asset substitution occurs optimally (default option gets more value if volatility is higher)
- With asset substitution, the optimal maturity is shorter, consistent with the idea that short-term debt helps curb agency problems (numerical result, not sure robust)
- Quantitatively, agency cost due to asset substitution is small
Leland (1998) (2)

- Assume threshold strategy that there exists $\delta_S$ s.t.
  \[ \sigma = \sigma_H \text{ for } \delta < \delta_S \text{ and } \sigma = \sigma_L \text{ for } \delta \geq \delta_S \]

- Solve for equity, debt, DTS, BC the same way as before, with one important change

- Need to piece solutions on $[\delta_B, \delta_S)$ and $[\delta_S, \infty)$ together

- $-\gamma_H, \eta_H, -\gamma_L, \eta_L$: solutions to fundamental quadratic equations
  \[
  D^H(\delta) = \frac{C}{r} + K^H_{\gamma} \delta^{1-\gamma_H} + K^H_{\eta} \delta^{\eta_H} \text{ for } [\delta_B, \delta_S)
  
  D^L(\delta) = \frac{C}{r} + K^L_{\gamma} \delta^{1-\gamma_L} + K^L_{\eta} \delta^{\eta_L} \text{ for } [\delta_S, \infty)
  
  \]

- Four boundary conditions to get $K^H_{\gamma}, K^H_{\eta}, K^L_{\gamma}, K^L_{\eta}$

- $K^L_{\eta} = 0$ because $D(\delta = \infty) < \frac{C}{r}$. The other three:
  \[
  D^H(\delta_S) = D^L(\delta_S) \text{ (value matching)}, \quad D^{H'}(\delta_S) = D^{L'}(\delta_S) \text{ (smooth pasting)}, \quad D^H(\delta_B) = \frac{(1-\alpha)\delta_B}{r-\mu} \text{ (value matching)}
  
  \]

- Here, smooth pasting at $\delta_S$ always holds, because Brownian crosses $\delta_S$ "super" fast. The process does not stop there (like at $\delta_B$)
Leland and Toft (1996)

- Deterministic maturity, but keep uniform distribution of debt maturity structure
- Say we have debts with a total measure of 1, maturity is uniformly distributed $U[0, T]$, same principal $P$, same coupon $C$
- Tough: now debt price is $D(\delta, t)$, need to solve a PDE
- Equity promises to keep the same maturity structure in the future
- Equity holders’ cashflows are

$$
\delta_t dt - (1 - \tau) Cdt - \frac{1}{T} dt \left( P - D(\delta, T) \right)
$$

- Cashflows $\delta_t dt$; Coupon $Cdt$; and Rollover losses/gains
- Over $[t, t + dt]$, there is $\frac{1}{T} dt$ measure of debt matures, equity holders need to pay

$$
\frac{1}{T} dt \left( P - D(\delta, T) \right)
$$

as equity holders get $D(\delta, T) \frac{1}{T} dt$ by issuing new debt
Leland and Toft (1996)

- First step: solve the PDE

\[ rD(\delta, t) = C + D_t(\delta, t) + \mu \delta D_\delta(\delta, t) + \frac{1}{2} \sigma^2 \delta^2 D_{\delta \delta}(\delta, t) \]

Boundary conditions

\[ D(\delta = \infty, t) = \frac{C}{r} (1 - e^{-rt}) + Pe^{-rt}: \text{defaultless bond} \]

\[ D(\delta = \delta_B, t) = (1 - \alpha) \frac{\delta_B}{r - \mu}: \text{defaulted bond} \]

\[ D(\delta, 0) = P \text{ for } \delta \geq \delta_B: \text{paid back in full when it matures} \]

- Leland-Toft (1996) get closed-form solutions for debt values; have a look

- Better know the counterpart of Feyman-Kac formula. The point is to know it admits closed-form solution
Leland and Toft (1996)

- Equity value satisfies the ODE

\[ rE(\delta) = \delta - (1 - \tau) C + \frac{1}{T} [D(\delta, T) - P] + \mu \delta E_\delta(\delta) + \frac{1}{2} \sigma^2 \delta^2 E_{\delta\delta}(\delta) \]

- This is also very tough, given the complicated form of \( D(\delta, T) \)!

- Leland and Toft have a trick (Modigliani-Miller idea): \( E(\delta) = \)

\[ v(\delta) - \frac{1}{T} \int_0^T D(\delta, t) \, dt = \frac{\delta}{r - \mu} + DTS(\delta) - BC(\delta) - \frac{1}{T} \int_0^T D(\delta, t) \, dt \]

- \( DTS(\delta) \) and \( BC(\delta) \) are much easier to price
  - \( DTS(\delta) \) is the value for constant flow payoff \( \tau C \) till default occurs
  - \( BC(\delta) \) is the value of bankruptcy cost incurred on default
  - We have derived them given \( \delta_B \)

- After getting \( E(\delta; \delta_B) \), \( \delta_B \) is determined by smooth pasting \( E'(\delta_B; \delta_B) = 0 \)

- In He-Xiong (2012), we introduce market trading frictions for corporate bonds
  - Some deadweight loss during trading, the above trick does not work
Calculation of Debt Tax Shield

Let us price $DTS (\delta)$ which is the value for constant flow payoff $\tau C$ till default occurs.

We can have

$$DTS (\delta) = \mathbb{E} \left[ \int_0^{\tau_B} e^{-rs} \tau C ds \right]$$

$$= \mathbb{E} \left[ \frac{\tau C}{r} (1 - e^{-r \tau_B}) \right] = \frac{\tau C}{r} \left( 1 - \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \right)$$

Or, $F (\delta) = DTS (\delta)$

$$rF (\delta) = \tau C + \mu \delta F_\delta (\delta) + \frac{1}{2} \sigma^2 \delta^2 F_\delta (\delta)$$

$$F (\delta) = \frac{\tau C}{r} + K \gamma \delta^{-\gamma} + K \eta \delta^\eta$$

Plugging $F (\delta_B) = 0$ and $F (\infty) = \frac{\tau C}{r}$ (so $K \eta = 0$) we have

$$F (\delta) = \frac{\tau C}{r} \left( 1 - \left( \frac{\delta}{\delta_B} \right)^{-\gamma} \right)$$
How to model negotiation and strategic debt service?

Consider a firm producing one widget per unit of time, random widget price

\[
dp_t / p_t = \mu dt + \sigma dZ_t
\]

Constant production cost \( w > 0 \) so cash flows are \( p_t - w \)

If debt holders come in to manage the firm, cash flows are \( \xi_1 p_t - \xi_0 w \) with \( \xi_1 < 1 \) and \( \xi_0 > 1 \)

Even without debt, \( p_t \) can be so low that shutting down the firm is optimal

This is so called "operating leverage"

One explanation for why Leland models predict too high leverage relative to data: Leland model includes operating leverage

For debt holders, if they take over, value is \( X(p) \) (need to figure out their hypothetical optimal stopping time by using smooth-pasting condition)
Now imagine the original coupon is $b > 0$

When $p_t$ goes down, what if equity holders can make a take-it-or-leave-it offer to debt holders?

Denote the equilibrium coupon service $s(p)$, and resulting debt value $L(p)$

In equilibrium there exist two thresholds $p_c < p_s$

- When $p_t \geq p_s$, $s(p) = b$, nothing happens
- When $p_t \in (p_c, p_s)$, we have $s(p) < b$ and $L(p) = X(p)$. As long as debt service is less than the contracted coupon, the value of debt equals that of debtholders’ outside option $X(p)$
- When $p_t$ hits $p_c$, liquidating the firm

When $s(p) < b$ we have $s(p) = \xi_1 p_t - \xi_0 w$ which is as if debt holders take the firm.

- In the paper, there is some complication of $\gamma > 0$ which is the firm’s scrap value
Miao, Hackbarth, Morellec (2006)

- Firm EBIT is $y_t \delta_t$, $y_t$ aggregate business cycle condition

$$d\delta_t/\delta_t = \mu dt + \sigma dZ_t$$

$y_t \in \{y_G, y_B\}$: Markov Chain

- Exponentially decaying debt, etc, same as Leland (1998)
- Default boundary depends on the current macro state: $\delta^G_B$ and $\delta^B_B$. Same smooth-pasting condition
- $\delta^G_B < \delta^B_B$, default more in $B$. Help explain credit spread puzzle
  - Bond seems too cheap in the data. If bond payoff is lower in recession, then it requires a higher return
- Lots of papers about credit spread puzzle use this framework

$$d\delta_t/\delta_t = \mu_s dt + \sigma_s dZ_t$$

where $s \in \{G, B\}$ or more

- ODE in vector: $x = \ln(\delta)$, $\mathbf{D}(x) = \begin{bmatrix} D^G(x) & D^B(x) \end{bmatrix}'$

$$r\mathbf{D}(x) = c\mathbf{1}_{2 \times 1} + \mu_{2 \times 2} \mathbf{D}'(x) + \frac{1}{2} \Sigma_{2 \times 2} \mathbf{D}''(x)$$

see my recent Chen, Cui, He, Milbradt (2014) if you are interested