Lecture 3B: Macroeconomic Models with Financial Intermediaries

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Introduction

- Theoretical papers
  - Holmstrom and Tirole (1997 QJE): two period model
  - He and Krishnamurthy (2012 Restud, 2013 AER): dynamic (continuous-time) model, better connecting to canonical finance framework
  - Brunnermeier and Sannikov (2014, AER): dynamic (continuous-time) model, better connecting to macro framework

- Connection to traditional empirical asset pricing literature
  - Adrian-Etula-Muir (2014): broker-dealer leverage alone can price FF 25 portfolios
  - He-Kelly-Manela (2017): Primary Dealer (market) capital ratio factor can consistently price a wide range of asset classes, including sophisticated ones like Options and CDS
A Model of Capital and Crises, 2012

- A General Equilibrium (GE) model where intermediaries, rather than households, are marginal.
  - Frictions are endogenously derived based on optimal contracting considerations. This affects prices.
  - Contracting takes future price dynamics into consideration.

- Mechanism: Intermediation capital affects participation/risk-sharing.
- In normal times households participate through intermediation;
- When intermediaries suffer losses,
  - Households “fly” away from intermediaries to riskless assets, driving down interest rate.
  - Distressed intermediary sector averse to hold risky positions, risk premium goes up.
Unit supply of risky asset with dividend \( \frac{dD_t}{Dt} = gd_t + \sigma dZ_t \), and riskless asset in zero-net supply.

- Risky asset price \( P_t \) and interest rate \( r_t \) are determined in GE.

Households \( \mathbb{E} \left[ \int_0^\infty e^{-\rho h t} \ln c_t^h dt \right] \).

- Limited participation in risky asset market. They invest in intermediaries.

Specialists \( \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \ln c_t dt \right] \), \( \rho < \rho^h \). They run intermediaries.

- Only intermediaries/specialists can invest in the risky asset. They are marginal investors.
- Derive Intermediation Constraint from moral hazard primitives.
- **Intermediation**: 1) Short-term contracting between agents; 2) Equilibrium in competitive intermediation market;
  - No friction in short-term-borrowing/repo market.
- **Asset pricing**: 3) Optimal consumption/portfolio decisions; 4) GE.
The Heart of the Model: Capital Constraint

- Say household with wealth $W_t^h$, and specialist with wealth $W_t$.
  - Given specialist’s equity contribution $W_t$ in the intermediary, household contributes $T_t^h$ as equity investment.
  - Capital Constraint: $T_t^h$ is capped at $mW_t$.

- Intermediation capacity $mW_t$ is increasing in the specialist's contribution $W_t$, as reflection of agency friction.

- How to interpret $m$?
  1. Intermediary capital requirement: outside/inside contribution ratio; (Holmstrom-Tirole, QJE)
     - Officers/Directors inside holdings in financial industry around 18%.
  2. Incentive contract—the performance share of hedge fund managers. Think of “2 and 20.”
  3. Mutual funds’ flow-performance sensitivity. Specialist’s $W_t$ tracks his past gains and losses (Shleifer-Vishny, JF, Limits to Arbitrage)
Intermediation Constraint: An Example

- Say $m = 1$, $W_t^h = 80$. Comparing $W_t^h$ to $mW_t$.

- **Unconstrained Region:** $W_t = 100$. Then $T_t^h = W_t^h = 80$;
  - Zero net debt. Risky asset price $P_t = W_t + W_t^h = 180$.
  - Fund’s total equity 180. Intermediary holds risky asset without leverage, first-best risk sharing.

- **Constrained Region:** $W_t = 50$. Then $T_t^h = mW_t = 50$;
  - Intermediary’s total equity is $50 + 50 = 100$. But $P_t = 130$.
  - In equilibrium, the intermediary borrows 30 from the debt market;
    - It is supplied by households $W_t^h - T_t^h = 30$.
  - Specialist and household have equal shares in the intermediary;
  - Specialist’s leveraged position in risky asset: $\alpha = \frac{50+15}{50} = 130\%$.
  - Risk premium has to adjust to make this high leverage optimal.
Risk Premium and Interest Rate

\[ w_c^{(m=4)} = 13.02 \]
\[ w_c^{(m=6)} = 9.07 \]

\[ w^{(m=6)} = 0.07 \]
\[ w^{(m=4)} = 13.02 \]
Cyclicality of Leverage across Intermediaries

Ang, Gorovvy and Van-Inwegen (2010)

- Commercial banking sector, market leverage is strongly countercyclical, but book leverage is procyclical
Primary Dealers’ Capital Ratio

- Primary Dealers (designated by NYFed to implement monetary policy); constructing capital ratio

$$\frac{\sum_i \text{MarketEquity}_{it}}{\sum_i (\text{MarketEquity}_{it} + \text{BookDebt}_{it})}$$
Road Map

- Intermediation contracts;
  - IC constraints, maximum exposure supply, etc.
- Agents’ consumption/portfolio decisions;
- Competitive equilibrium in intermediation markets;
- Equilibrium asset prices.
- Conclusion.
Intermediation Stage Game

- **Short-term** contracts only. At time $t$, contract from $t$ to $t + dt$.

- Household with wealth $W^h_t$, and specialist with wealth $W_t$.
  - Household contributes $T^h_t$, specialist $T_t$. $T^l_t = T^h_t + T_t$.

- Specialist in charge of intermediary. **Moral Hazard:**
  1. Unobserved due diligence action $s_t = \{0, 1\}$.
     - Shirking ($s_t = 1$) reduce return by $X_t$ but brings private benefit $B_t$.
  2. Unobserved portfolio choice $E^l_t$ (dollar exposure to risky asset);
     - Undoing activity. Not crucial.

- Fund's return $E^l_t (dR_t - r_t dt) + T^l_t r_t dt - s_t X_t dt$, private benefit
  $s_t B_t dt$. Focus on implementing working.
  - Risky asset return $dR_t = \frac{dP_t + D_t dt}{P_t}$ and interest rate $r_t$ are endogenous.
Intermediation Contract

- **Affine contracts** for sharing returns.
  - $\beta_t$: specialist’s share; $\hat{K}_t\,dt$: transfer to specialist.

- $\Pi_t \equiv \left( T_t, T^h_t, \beta_t, \hat{K}_t \right) \in [0, W_t] \times [0, W^h_t] \times [0, 1] \times \mathbb{R}$.

- Define $K_t \equiv \left( \beta_t T^l_t - T_t \right) r_t + \hat{K}_t$.

- **Dynamic budget constraint**

  \[
  \begin{aligned}
  dW_t &= W_t r_t \, dt - c_t \, dt + \beta_t \mathcal{E}^l_t \left( dR_t - r_t \, dt \right) + K_t \, dt, \\
  dW^h_t &= W^h_t r_t \, dt - c^h_t \, dt + (1 - \beta_t) \mathcal{E}^l_t \left( dR_t - r_t \, dt \right) - K_t \, dt.
  \end{aligned}
  \]

- Reduce contract to $(\beta_t, K_t)$. **Sharing rule** and **fee**.
  - Specialist chooses $\mathcal{E}_t = \beta_t \mathcal{E}^l_t$. Household buys risk exposure $\mathcal{E}^h_t = (1 - \beta_t) \mathcal{E}^l_t$ from intermediary.
  - In competitive intermediation market, the fee will take some simple linear form.
IC Constraint and Maximum Household’s Exposure

- \( \mathcal{E}_t^l \) fund’s total risk exposure. S: \( \mathcal{E}_t = \beta_t \mathcal{E}_t^l \), H: \( \mathcal{E}_t^h = (1 - \beta_t) \mathcal{E}_t^l \).
- Specialist net worth

\[
dW_t = W_t r_t dt - c_t dt + \beta_t \mathcal{E}_t^l (dR_t - r_t dt) + K_t dt + s_t (B_t - \beta_t X_t) dt
\]

- IC constraint: a lower bound on \( \beta_t \). Skin in the game
  - No shirking: \( \beta_t X_t - B_t \geq 0 \Rightarrow \beta_t \geq \frac{B_t}{X_t} = \frac{1}{1+m} \).
- Specialist always chooses \( \beta_t \mathcal{E}_t^l = \mathcal{E}_t^* \) independent of \( \beta_t \).
  - In the paper we show \( \mathcal{E}_t^* \) is independent of fee \( K \).
- Household exposure from the contract

\[
\mathcal{E}_t^h = (1 - \beta_t) \mathcal{E}_t^l = \frac{1 - \beta_t}{\beta_t} \mathcal{E}_t^*.
\]

- Plugging \( \beta_t \geq \frac{1}{1+m} \), household risk exposure \( \mathcal{E}_t^h \leq m \mathcal{E}_t^* \).
Key Intuition and Equity Implementation

- The households risk exposure is capped due to agency frictions

\[ \mathcal{E}_t^h \leq m \mathcal{E}_t^* . \]

- Cap on how to share the aggregate risk \( \mathcal{E}_t^h + \mathcal{E}_t^* = \mathcal{E}_t^l \) between households and specialists.
- Incentive provision implies that specialists have to bear sufficient risk.

- In bad times this friction kicks in.
  - Even if specialists net worth is low, they still have to bear large risk.

- **Equity implementation**: Households (outsiders) cannot hold more than \( \frac{m}{1+m} \) (equity) shares.

- **Equity capital constraint**: Given specialist’s equity \( W_t \), households can make at most \( mW_t \) equity contributions.

- Recall contract \( (\beta_t, K_t) \). We have derived equilibrium \( \beta_t \). What determines fee \( K_t \)?
  - Households pay competitive fees in the intermediation market.
**Competitive Intermediation Market**

**Definition.** At time $t$, specialists make offers $(\beta_t, K_t)$ to specialists; households can accept/reject the offers. The intermediation market reaches equilibrium if:

1. $\beta_t$ is incentive compatible for each specialist.
2. There is no coalition of households and specialists, such that some incentive-compatible contracts can make households strictly better off while specialists weakly better off.

**Lemma 4:** Given symmetry at the beginning of time-$t$, the resulting intermediation equilibrium is symmetric.

**Lemma 5:** Households face an equilibrium per-unit-exposure price of $k_t \geq 0$: to purchase $\mathcal{E}_t^h$, he has to pay $K_t = k_t \mathcal{E}_t^h$.

- Idea: equivalence between **Core** and **Walrasian equilibrium**; coalition can chop off exposure in a linear way.
Households’ Consumption/Portfolio Rules

- Log investors. Simple consumption rule; myopic mean-variance portfolio choice.

- Risky asset return \( dR_t = \left( \pi_{R,t} + r_t \right) dt + \sigma_{R,t} dZ_t \).
  - \( \pi_{R,t} \) is risk premium, \( \frac{\pi_{R,t}}{\sigma_{R,t}} \) is the so-called Sharpe ratio

- Household

\[
\text{max} \quad \mathbb{E} \left[ \int_0^\infty e^{-\rho h t} \ln c_t^h \, dt \right] \quad \text{subject to}
\]

\[
dW_t^h = W_t^h r_t \, dt - c_t^h \, dt + \mathcal{E}_t^h \left( dR_t - r_t \, dt \right) - k_t \mathcal{E}_t^h \, dt.
\]

- Relative to standard problem, households achieve exposure \( \mathcal{E}_t^h \) by paying per-unit-cost of \( k_t \).

- Optimal consumption \( c_t^{h^*} = \rho^h W_t^h \), optimal exposure \( \mathcal{E}_t^{h^*} = \frac{\pi_{R,t} - k_t}{\sigma^2_{R,t}} W_t^h \).
  - Optimal risk exposure is decreasing in exposure price \( k_t \).
Specialists’ Consumption/Portfolio Rules

- Specialist supplies exposure $\frac{1-\beta_t}{\beta_t} \mathcal{E}_t^*$. Given exposure price $k_t$, he gets intermediation fees $K_t dt = k_t \left( \frac{1-\beta_t}{\beta_t} \mathcal{E}_t^* \right) dt$.

- The specialist is solving: $\max_{\{c_t, \mathcal{E}_t, \beta_t\}} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \ln c_t dt \right]$ subject to

$$dW_t = \mathcal{E}_t (dR_t - r_t dt) + W_t r_t dt - c_t dt + \max_{\beta_t \in \left[ \frac{1}{1+m}, 1 \right]} \left( \frac{1-\beta_t}{\beta_t} \right) \mathcal{E}_t^* k_t dt.$$

- **Exposure supply schedule:** $\beta_t^* = \frac{1}{1+m}$ if $k_t > 0$, otherwise $\beta_t^* \in \left[ \frac{1}{1+m}, 1 \right]$ if $k_t = 0$.

- Solution: $c_t^* = \rho W_t$ and $\mathcal{E}_t^* = \frac{\pi R_t}{\sigma^2_{R,t}} W_t$.

- Total exposure fee is linear in $W_t$, as if getting a better return from their wealth. Traditional log-agent results apply.
Equilibrium in Competitive Intermediation Market

- Exposure demand $E^h_t (k_t) = \frac{\pi_{R,t} - k_t}{\sigma^2_{R,t}} W^h_t$; exposure supply is free but with maximum $mE^*_t = m\frac{\pi_{R,t}}{\sigma^2_{R,t}} W_t$.

  ▶ As intermediary portfolio choice is unobservable, maximum exposure supply is determined only by specialist’s wealth and asset prices
  ▶ We also solve the case with observable portfolio choice, so supply is increasing with $k_t$

**Proposition 1:** The economy is in one of two equilibria:

1. Unconstrained region $E^h_t (k_t = 0) < mE^*_t$, and $\beta_t < \frac{1}{1+m}$.
   ▶ Excess intermediation supply, zero rent.

2. Constrained region $E^h_t (k_t > 0) = mE^*_t$, and $\beta_t = \frac{1}{1+m}$.
   ▶ Scarce intermediation supply, positive rent.
Unconstrained vs. Constrained Regions (1)

Unconstrained Region

Exposure demand:

\[
\left( \frac{\pi_{R,t} - k_t}{\sigma_{R,t}^2} \right) W_t^k
\]

Exposure supply:

\[
\begin{cases} 
0, \frac{m \left( \frac{\pi_{R,t}}{\sigma_{R,t}^2} \right) W_t}{m \left( \frac{\pi_{R,t}}{\sigma_{R,t}^2} \right) W_t} 
\text{if } k_t = 0, \\
\frac{m \left( \frac{\pi_{R,t}}{\sigma_{R,t}^2} \right) W_t}{m \left( \frac{\pi_{R,t}}{\sigma_{R,t}^2} \right) W_t} 
\text{if } k_t > 0.
\end{cases}
\]
Unconstrained vs. Constrained Regions (1)

Constrained Region

price $k_t$

Exposure supply:
\[
\begin{cases}
0, m \left( \frac{\pi_R}{\sigma_R^2} \right) W_t & \text{if } k_t = 0, \\
m \left( \frac{\pi_R}{\sigma_R^2} \right) W_t & \text{if } k_t > 0.
\end{cases}
\]

Exposure demand:
\[
\left( \frac{\pi_{R_t} - k_t}{\sigma_{R_t}^2} \right) W_t^k
\]
Equilibrium Asset Prices: Solution

- We derive everything in closed form.
- State variables \((D_t, W_t)\). Scales with \(D_t\).
- Uni-dimensional state variable \(w_t \equiv W_t / D_t\) captures wealth distribution.
- Consumption rules \(c_t^* = \rho W_t^h\), \(c_t^{h*} = \rho^h W_t^h\).
- Zero net debt \(W_t + W_t^h = P_t\), goods clearing \(c_t^* + c_t^{h*} = D_t\). So
  \[
  \frac{P_t}{D_t} = \frac{1}{\rho^h} + \left(1 - \frac{\rho}{\rho^h}\right) w_t.
  \]
- Specialist’s risky position \(\alpha_t = \frac{P_t}{(1+m)W_t} > 1\) in constrained region.
The economy is in constrained region whenever

\[ w_t = \frac{W_t}{D_t} < w^c \equiv \frac{1}{m\rho^h + \rho}. \]

- Unconstrained region, \( w_t \) moves deterministically. Constrained region, specialists take a higher leverage than households, so \( w_t \) drops when fundamental falls.

- When intermediary capital \( W_t \) falls,
  - Risk premium rises;
  - Interest rate falls;
  - Volatility rises;
  - Correlation endogenously rises.
\( \alpha_t\): intermediary's risk position Asset/Equity

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<th>Uncon. Region</th>
<th>Con. Region</th>
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<tbody>
<tr>
<td>( \alpha_t )</td>
<td>1</td>
<td>( \frac{1 + (\rho^h - \rho)w_t}{(1+m)\rho^h w_t} &gt; 1 )</td>
</tr>
<tr>
<td>( \sigma_{R,t} )</td>
<td>( \sigma )</td>
<td>( \frac{\sigma}{1 + (\rho^h - \rho)w_t} \left( \frac{(1+m)\rho^h}{m\rho^h + \rho} \right) &gt; \sigma )</td>
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<tr>
<td>( \pi_{R,t} )</td>
<td>( \sigma^2 )</td>
<td>( \frac{\sigma^2}{w_t(m\rho^h + \rho)} \left( \frac{(1+m)\rho^h}{m\rho^h + \rho} \right) \left( \frac{1}{1 + (\rho^h - \rho)w_t} \right) &gt; \sigma^2 )</td>
</tr>
<tr>
<td>( r_t )</td>
<td>( \rho^h + g - \sigma^2 + \rho \left( \rho - \rho^h \right) w_t )</td>
<td>( -\sigma^2 \left[ \frac{\rho \left( (1+m) \left( \frac{1}{w_t} - \rho \right) - m^2 \rho^h \right) w_t + (m\rho^h)^2}{(1-\rho w_t)(\rho + m\rho^h)^2} \right] )</td>
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</table>
- Asymmetry. Crisis like.
- When constraint binds $w_t < w^c$, specialist bears disproportionately large risk, causing more volatile pricing kernel.
- Flight to quality. 1) Specialists precautionary savings. 2) Household fly to debt market.
Concluding Remarks

- Canonical intermediation friction meets canonical GE asset pricing models.
- Calibratable, easy to quantify effects.
- We have another paper where specialists have general CRRA power utility, with capital constraint as given.
  - Add in labor income, debt households (create leverage in unconstrained region), and other necessary twists...
  - Study the crisis dynamics (especially recovery), government liquidity injection policies, etc.
Intermediary Asset Pricing (2013)

Specialists
(1) Portfolio choice for intermediary
(2) Wealth = \( W_t \)
(3) Capital constraint

HOUSEHOLDS

OLG structure
(1) Labor income
(2) Consume
(3) Save (portfolio choice) to leave bequest
(4) Wealth = \( W_t^n \)
(5) Minimum of \( \lambda W_t^b \) in bond

H_\text{t}
EQUITY

DEBT

SPECIALISTS/INTERMEDIARIES

RISKY ASSET MARKET

RISKLESS DEBT
(intermediary repo)
Model

- Risky asset (unit supply) with dividend
  \[ \frac{dD_t}{D_t} = g dt + \sigma dZ_t; \]

- Labor income \( lD_t dt \), so the total endowment is
  \[ (1 + l) D_t dt \]

- Price/dividend ratio \( p_t = P_t / D_t \) (so \( dR_t = (D_t dt + dP_t) / P_t \)) and interest rate \( r_t \) to be solved

- \( D \) scales the economy, "wealth distribution" key state variable

- State variable in the paper: the fraction of specialist’s wealth
  \[ x_t = W_t / P_t \]
  - It is \( \eta_t \) in Brunnermeier-Sannikov
  - We are solving for \( p(x) \) and \( r(x) \)
Assumptions

- Households wealth $W^h$ and specialist wealth $W$, with $W + W^h = P$
- **Assumption:** $\lambda$ fraction of household wealth, i.e. $\lambda W^h$ is placed in debt, supplied by intermediaries
- **Assumption:** Equity household in charge of $(1 - \lambda) W^h_t$, choosing $\alpha^h_t \in [0, 1]$ to put in intermediaries with constraint
  \[ \alpha^h_t (1 - \lambda) W^h_t \equiv H_t \leq mW_t \]
- Intermediaries are managed by specialists $\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt \right]$
- Specialists: put their own wealth in as equity; get households equity capital; choose $\alpha^l_t$ for the intermediary so
  \[ \widetilde{dR}_t = r_t dt + \alpha^l_t (dR_t - r_t dt) \]
- Households wealth evolution
  \[
  dW^h_t = (lD_t - \rho W^h_t) dt + W^h_t r_t dt + \alpha^h_t (1 - \lambda) W^h_t \left( \widetilde{dR}_t - r_t dt \right) \\
  = (lD_t - \rho W^h_t) dt + W^h_t r_t dt + \alpha^h_t (1 - \lambda) W^h_t \alpha^l_t (dR_t - r_t dt)
  \]
**Steps of Deriving Equilibrium (1)**

- Specialist is marginal investor. Faces no constraints. Portfolio choice must be an optimal choice for him.
- His Euler equation is always valid. Household is always constrained.
- Easier to trace households financial wealth scaled by dividend

\[
y_t = \frac{P_t - W_t}{D_t} = \frac{P_t - W_t}{P_t} \frac{1}{p_t} = \frac{1 - w_t}{p_t}
\]

- Log household implies \( c^h_t = \rho W^h_t = \rho y_t D_t \)
  - We need short-lived household who does not value future labor income.
- Goods market clearing implies

\[
c_t = (1 + l) D_t - c^h_t = (1 + l - \rho y_t) D_t
\]

- Euler equation for specialist (\( \Lambda_t = e^{-\rho t} c^{-\gamma} \))

\[
-\rho dt - \gamma E_t \left[ \frac{dc_t}{c_t} \right] + \frac{1}{2} \gamma(\gamma + 1) \text{Var}_t \left[ \frac{dc_t}{c_t} \right] + \mathbb{E}_t [dR_t] = \gamma \text{Cov}_t \left[ \frac{dc_t}{c_t}, dR_t \right]
\]

\[
\rho dt - \gamma \mathbb{E}_t \left[ \frac{dc_t}{c_t} \right] + \frac{\gamma(\gamma + 1)}{2} \text{Var}_t \left[ \frac{dc_t}{c_t} \right] + r_t dt = 0
\]
Steps of Deriving Equilibrium (2)

- Denote P/D ratio $F(y) \equiv p(x)$
- **Goal:** derive the equilibrium evolution of $y_t$ and hence $c_t$
- Say $dy_t = \mu_y dt + \sigma_y dZ_t$ where $\mu_y$ and $\sigma_y$ to be solved
- Households’ effective portfolio exposure to risky asset

\[ \theta_S(y_t) = \alpha^l_t \alpha^h_t (1 - \lambda) \]

- Budget equation for households implies

\[
D_t \left[ (\mu_y dt + \sigma_y dZ_t) + y_t (g dt + \sigma dZ_t) + \sigma_y \sigma dt \right] \overset{\text{Ito's lemma}}{=} d (y_t D_t) = dW^h_t
\]

\[ \overset{\text{Budget Equation}}{=} (ID_t - \rho W^h_t) dt + W^h_t r_t dt + \alpha^l_t \alpha^h_t (1 - \lambda) W^h_t (dR_t - r_t dt) \]

- Divide both sides by $D_t$ we have

\[
\left( \mu_y + gy_t + \sigma_y \sigma \right) dt + (\sigma_y + \sigma_y y_t) dZ_t
\]

\[ = \left( l - \rho y_t \right) dt + y_t r_t dt + \theta_S(y_t) y_t (dR_t - r_t dt) \]
Steps of Deriving Equilibrium (3)

- Write out $dR_t$ in terms of $F(y)$

$$dR_t = \frac{D_t dt + dP_t}{P_t} = \frac{1}{F(y)} dt + \frac{dP_t}{P_t}$$

where $dP_t = d \left( F(y_t) D_t \right)$ so

$$\frac{dP_t}{P_t} = g dt + \sigma dZ_t + \left[ \frac{F'(y_t)}{F(y_t)} \left( \mu_y dt + \sigma_y dZ_t \right) + \frac{F''(y_t)}{2F(y_t)} \sigma_y^2 dt \right] + \frac{\sigma F'(y_t) \sigma_y}{F(y_t)} dt$$

- Equating volatility terms:

$$\sigma_y + \sigma y_t = \theta_S (y_t) y_t \left[ \sigma + \frac{F'(y_t)}{F(y_t)} \sigma_y \right] \Rightarrow \sigma_y = \frac{\sigma y_t (\theta_S (y_t) - 1)}{1 - \theta_S (y_t) y_t \frac{F'(y_t)}{F(y_t)}}$$

- Drift term $\mu_y$ follows similarly but more tedious
Steps of Deriving Equilibrium (4)

- Now figure out household equilibrium risk exposure $\theta_S(y_t)$
- Focus on the parameterization where in equilibrium
  - Unconstrained region, $\alpha^h_t = 1$ and $(1 - \lambda)W^h_t \equiv H_t < mW_t$ (equity household wants to put more, but debt household says no)
  - Constrained region, $\alpha^h_t = \frac{mW_t}{(1-\lambda)W^h_t} < 1$
- Unconstrained region: $y < y_c$ abundant intermediary capital
  - $\alpha^h_t = 1$ and $\alpha^l_t = \frac{F(y)}{F(y) - \lambda y}$
  - Derive $\alpha^l_t$: risky asset is $F(y)D$, but intermediaries have equity of
    $\left( \underbrace{F(y) - y}_{\text{Specialists' wealth as equity}} + \underbrace{(1 - \lambda) y}_{\text{Equity households's equity}} \right) D = (F(y) - \lambda y)D$
  - Hence $\theta_S(y_t) = \alpha^l_t \alpha^h_t (1 - \lambda) = \frac{(1-\lambda)F(y)}{F(y) - \lambda y}$
- Constrained region: $y > y_c$, scarce intermediary capital
  - $\theta_S(y) = \frac{1}{1+m}$ (households and specialist share risk $1 : m$)
Steps of Deriving Equilibrium (5)

- With $\theta_S(y_t)$ expressed as functions of $F(y)$, we can derive $\mu_y$ and $\sigma_y$ in terms of $F(y)$, $F'(y)$ and $F''(y)$
- Recall Euler equation

$$-ho dt - \gamma \mathbb{E}_t \left[ \frac{dc_t}{c_t} \right] + \frac{1}{2} \gamma (\gamma + 1) \text{Var}_t \left[ \frac{dc_t}{c_t} \right] + \mathbb{E}_t [dR_t] = \gamma \text{Cov}_t \left[ \frac{dc_t}{c_t}, dR_t \right]$$

- Recall evolution of specialist’s consumption growth

$$\frac{dc_t}{c_t} = \frac{d \left( (1 + l - \rho y_t) D_t \right)}{(1 + l - \rho y_t) D_t}$$

- We can write down $dc_t / c_t$ in terms of $F(y)$, $F'(y)$ and $F''(y)$
- $\mathbb{E}_t [dR_t]$ are in terms of $F(y)$, $F'(y)$, and $F''(y)$, with $\mu_y$ and $\sigma_y$
- So, eventually we arrive at an ODE in $F$!
- Boundary conditions $y = 0$ (specialists dominating) and $y = \frac{1 + l}{\rho}$ (households dominating)
  - They are singular boundaries to be derived from ODE itself
Crisis Recovery

<table>
<thead>
<tr>
<th>Transit to</th>
<th>Transit from 12%</th>
<th>Incremental Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td>7.5%</td>
<td>0.65</td>
<td>0.47</td>
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<tr>
<td>5%</td>
<td>2.67</td>
<td>0.77</td>
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<tr>
<td>4%</td>
<td>5.56</td>
<td>2.90</td>
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</table>
Concluding Remarks

- We also did a bunch of policy experiment, like borrowing subsidy, Asset purchase, or TARP
- Some counterfactual predictions
  - Interest rate goes to pretty negative around -9% when risk premium is 12%
  - Volatility goes down eventually when risk premium goes to 12%
- There are no real side in this paper
- Working paper "A Macroeconomic Framework to Quantify Systemic Risk"
  - Include real side and housing
  - Model crisis as "occasionally binding constraint" and emphasize global solution method
    - In contrast to log-linearization around steady state
  - Document strong nonlinearity in the data, and perform a serious calibration