Lecture 2B: Models on Dynamic Bank Runs

Zhiguo He

Booth School of Business

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Valuation or Hamilton-Jacobi-Bellman (HJB) Equation (1)

- \( V(y) = \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} f(y_s) \, ds \mid y_t = y \right] \) s.t.
  \[ dy_t = \mu(y_t) \, dt + \sigma(y_t) \, dZ_t \]

- Discrete-time Bellman equation
  \[ V(y) = \frac{1}{1+r} \left( f(y) + \mathbb{E} \left[ V(y') \mid y \right] \right) \] s.t. \( y' = y + \mu(y) + \sigma(y) \varepsilon \)

- Continuous-time, \( V(y) \) can be written as
  
  \[
  V(y) = \mathbb{E}_t \left[ f(y_t) \, dt + \int_{t+dt}^\infty e^{-r(s-t)} f(y_s) \, ds \mid y_{t+dt} = y_t + \mu(y_t) \, dt + \sigma(y_t) \, dZ_t \right] \\
  = f(y) \, dt + e^{-r \cdot dt} \mathbb{E}_t \left[ \int_{t+dt}^\infty e^{-r(s-t-dt)} f(y_s) \, ds \mid y_{t+dt} = y_t + \mu(y_t) \, dt + \sigma(y_t) \, dZ_t \right] \\
  = f(y) \, dt + e^{-r \cdot dt} \mathbb{E}_t \left[ \mathbb{E}_{t+dt} \left( \int_{t+dt}^\infty e^{-r(s-t-dt)} f(y_s) \, ds \mid y_{t+dt} = y_t + \mu(y_t) \, dt + \sigma(y_t) \, dZ_t \right) \right] \\
  = f(y) \, dt + (1 - rdt) \mathbb{E}_t \left[ V(y_t + \mu(y) \, dt + \sigma(y) \, dZ_t) \right] \\
  = f(y) \, dt + (1 - rdt) \mathbb{E}_t \left[ V(y_t) + V'(y_t) \mu(y_t) \, dt + V'(y_t) \sigma(y_t) \, dZ_t + \frac{1}{2} V''(y_t) \sigma^2(y_t) \, dt \right] \\
  = f(y) \, dt + (1 - rdt) \left[ V(y) + V'(y) \mu(y) \, dt + \frac{1}{2} V''(y) \sigma^2(y) \, dt \right]
  \]
Valuation or Hamilton-Jacobi-Bellman (HJB) Equation (2)

- Expansion of RHS:

\[ V(y) = f(y) \, dt + (1 - rdt) \left[ V(y) + V'(y) \mu(y) \, dt + \frac{1}{2} V''(y) \sigma^2(y) \, dt \right] \]

\[ = f(y) \, dt + V(y) + V'(y) \mu(y) \, dt + \frac{1}{2} V''(y) \sigma^2(y) \, dt \]

\[ - rV(y) \, dt - rV'(y) \mu(y) (dt)^2 - r \frac{1}{2} V''(y) \sigma^2(y) (dt)^2 \]

- From higher to lower orders, until non-trivial identity
  - At order \( O(1) \), \( V(y) = V(y) \), trivial identity
  - At order \( O(dt) \), non–trivial identity

\[ 0 = \left[ f(y) + V'(y) \mu(y) + \frac{1}{2} V''(y) \sigma^2(y) - rV(y) \right] \, dt \]

- As a result, we have

\[ rV(y) = f(y) + V'(y) \mu(y) + \frac{1}{2} \sigma^2(y) V''(y) \]

required return flow (dividend) payoff local change of value function (capital gain, long-term payoffs)

- That is how I write down value functions for any process (later I will introduce jumps)
General Solution for GBM process with Linear Flow Payoffs

- In the GBM setting, the model is special because

\[ f(y) = a + by, \mu(y) = \mu y, \text{ and } \sigma(y) = \sigma y \]

- It is well known that the general solution to \( V(y) \) is

\[ V(y) = \frac{a}{r} + \frac{b}{r - \mu} y + K_\gamma y^{-\gamma} + K_\eta y^\eta \]

where the "power" parameters are given by

\[ -\gamma = - \frac{\mu - \frac{1}{2} \sigma^2 + \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} < 0, \]

\[ \eta = - \frac{\mu - \frac{1}{2} \sigma^2 - \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} > 1 \]

- The constants \( K_\gamma \) and \( K_\eta \) are determined by boundary conditions
Side Note: How Do You Get Those Two Power Parameters

- Those two power parameters — \( -\gamma \) and \( \eta \) are roots to the fundamental quadratic equations.
- Consider the homogenous ODE:

\[
rV(y) = \mu y V'(y) + \frac{1}{2} \sigma^2 y^2 V''(y)
\]

- Guess the \( V(y) = y^x \), then \( V'(y) = xy^{x-1} \) and \( V''(y) = x(x-1)y^{x-2} \)

\[
ry^x = \mu xy^x + \frac{1}{2} \sigma^2 x (x-1) y^x
\]

\[
r = \mu x + \frac{1}{2} \sigma^2 x (x-1)
\]

\[
0 = \frac{1}{2} \sigma^2 x^2 + \left( \mu - \frac{1}{2} \sigma^2 \right) x - r
\]

- \( -\gamma \) and \( \eta \) are the two roots of this equation.
Halmilton-Jacobi-Bellman Equation with Jumps

\[ \rho V(y) = f(y) + V'(y) \mu(y) + \frac{1}{2} \sigma^2(y) V''(y) \]

- required return
- flow (dividend) payoff
- local change of value function (capital gain, long-term payoffs)

▷ Introduce "control" \( x_t \), choosing \( x_t \) to maximize value

▷ HJB with optimization (local optimization is enough!) is

\[ \rho V(y) = \max_x f(y; x) + V'(y) \mu(y; x) + \frac{1}{2} \sigma^2(y; x) V''(y) \]

- required return
- flow (dividend) payoff
- local change of value function (capital gain, long-term payoffs)

▷ Merton problem, \( x = (c, \alpha) \): consumption \( c \) and portfolio \( \alpha \), affecting evolution of wealth \( W \) (state \( y \))

\[ f(W; c) = u(c), \quad dW_t / W_t = -c_t dt + rd t + \alpha_t ((\mu - r) dt + \sigma dZ_t) \]

▷ "Stopping" control and Poisson event (say intensity \( \phi \))

\[ \rho V(y) = \max_x f(y; x) + V'(y) \mu(y; x) + \frac{1}{2} \sigma^2(y; x) V''(y) \]

- required return
- flow (dividend) payoff
- local change of value function (capital gain, long-term payoffs)

\[ + \phi (V(y; x, \text{post Poisson}) - V(y)) \]

- local change of value function due to Poisson
Runs on Non-bank Financial Institutions

- Runs on the non-bank financial institutions was one of the main causes of the credit crisis of 2007-2008.
  - e.g., Bernanke (2008), Cox (2008), Geithner (2008), Brunnermeier (2009), Gorton (2008), Krishnamurthy (2009), and Shin (2009).

- The classic Diamond-Dybvig model on bank runs:
  - The simultaneous coordination problem among depositors leads to a self-fulfilling bank-run equilibrium.

- Global-games models of bank runs:
  - Depositors observe noisy private signals about bank fundamental: Rochet-Vives (2004) and Goldstein-Pauzner (2005)

- Many questions about debt runs involve time-varying fundamentals:
  - How does a firm’s asset price volatility affect its debt run risk?
  - Do credit lines mitigate runs?
  - Do longer debt maturities mitigate runs?
Dynamic Debt Runs

- A model with time-varying fundamental without noisy private signals:
  - Fundamental is time-varying and all creditors share the same public information about fundamental.
  - The firm uses a staggered debt structure, i.e., creditors make rollover decisions at different times.

- A unique threshold equilibrium:
  - each creditor rolls over or not based on current fundamental;
  - a “rat race” among creditors in choosing rollover thresholds.

- Results similar to static global-games models:
  - Severe runs on firms with weaker fundamentals, greater illiquidity.

- New results:
  - Higher volatility increases strategic uncertainty and thus exacerbates runs.
  - When fundamental volatility is sufficiently high, stronger credit lines and longer debt maturities can exacerbate runs.
A Brief Literature Review

- **Static global games models:**
  - Rochet and Vives (2004), Goldstein and Pauzner (2005)

- **Dynamic coordination games based on higher order beliefs:**

- **Dynamic coordination games based on observable fundamentals:**
  - Frankel and Pauzner (2000)

- **Growing literature on modeling rollover risk:**
  - Acharya, Gale, and Yorulmzer (2011, JF), Morris and Shin (2009), Brunnermeier and Oehmke (2013, JF), He and Xiong (2012, JF)
The Model Structure

- A firm holds a long-term risky asset by rolling over short-term debt.
- The environment of illiquid/imperfect capital markets:
  - The firm cannot find a single creditor (with deep pockets) to finance all the debt, and has to rely on a continuum of small creditors.
  - When some creditors choose to run, the firm needs to draw on unreliable credit lines.
  - The firm asset is illiquid, i.e., the firm can only recover a fraction of its fundamental value in a premature liquidation.
- Two key assumptions:
  - The asset fundamental is time-varying and publicly observable
  - A staggered debt structure: coordination problem between creditors maturing at different times
The firm holds a long-term asset:

- The asset generates constant cash flow $r dt$ over a period $dt$.
- At a Poisson arrival time $\tau_\phi$ with intensity $\phi$, the asset matures with a final payoff equal to $\tau_\phi$ value of a publicly observable process:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t.$$  

Why Poisson? Memoryless, time elapsed does not matter.

Risk-neutral agents with discount rate $\rho$. Asset fundamental value:

$$F(y_t) = \mathbb{E}_t \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} rds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t$$

$$\rho F(y) = r + \mu y F'(y) + \frac{\sigma^2}{2} y^2 F''(y) + \phi (y - F(y)) \quad \text{HJB equation}$$

- $y_t$ is the firm fundamental, hit by persistent shocks.
- Our model ignores complications from private information.
Staggered Debt Financing

- Staggered structure and debt contract are exogenously given.
- A unit measure of short-term creditors (discount rate $\rho$).
  - Coupon flow payment $rdt$ over $(t, t + dt)$.
  - $r > \rho$.
- A staggered debt structure:
  - Each contract matures with a probability of $\delta dt$, a la Calvo (1983).
  - In aggregate, $\delta dt$ fraction of debt matures over $(t, t + dt)$.
  - This fraction is small and thus avoiding the Diamond-Dybvig type simultaneous coordination problem.
  - Key: during a contract lock-in period (say 3-month), other creditors might run.
- At $\tau_\delta$, an individual creditor decides to run or roll over.
  - Threshold strategy $y_\ast$: roll over if and only if $y \geq y_\ast$.
  - Assumption: only consider threshold strategies.
Debt Run and Liquidation

- Over \((t, t + dt)\), \(\delta dt\) fraction of contracts matures.
- If they choose to run, the firm needs to draw on its credit lines.
  - With prob \(\theta \delta dt\), the credit lines fail, causing the firm to fail.
    - \(\theta\): unreliability of credit lines, or uncertainty of cash reserves.
    - Can also be interpreted as imperfect government bailout.
  - With prob \(1 - \theta \delta dt\), the firm raises new fund and pays the creditors.
    - New debt takes the same contract form.
- Early liquidation recovers \(\alpha \in (0, 1)\) of the fundamental value:
  \[
  \widetilde{L}(y_t) = \alpha F(y_t).
  \]
- Liquidation decision is irreversible, no partial liquidation.
- The firm’s liquidation value, \(\widetilde{L}(y)\), is equally divided among all creditors, including the running ones.
- The probability of firm failing by one’s own run is tiny \((\theta \delta dt)\) ⇒ expected payoff from running is
  \[
  \theta \delta dt \cdot \alpha F(y_t) + (1 - \theta \delta dt) \cdot 1 = 1
  \]
Three Possible Paths for An Individual Creditor

- A creditor receives $r$ until a random time $\tau = \min(\tau_\phi, \tau_\delta, \tau_\theta)$;
- Other creditors’ rollover threshold $y_*$: rollover when $y > y_*$, run otherwise.

\[\begin{align*}
\tau &= \tau_\phi, \text{ asset maturity with } \\
&\quad \min(1, y) \text{ to the creditor} \\
\tau &= \tau_\delta, \text{ the contract expires for a rollover} \\
\tau &= \tau_\theta, \text{ firm bankruptcy with } \\
&\quad \min(1, L(y)) \text{ to the creditor}
\end{align*}\]
An Individual Creditor’s Problem

- Given other creditors’ threshold $y_*$, his value function is

\[
V(y_t; y_*) = \mathbb{E}_t \left\{ \int_t^\tau e^{-\rho(s-t)} r ds \\
+ e^{-\rho(\tau-t)} \min(1, y_\tau) 1_{\{\tau=\tau_\phi\}} \right\}
\]

Top path, the asset matures and pays off

\[
+ e^{-\rho(\tau-t)} \max \{1, V(y_\tau; y_*)\} 1_{\{\tau=\tau_\delta\}}
\]

Middle path, make the rollover decision when contract expires

\[
+ e^{-\rho(\tau-t)} \min(1, L + ly_\tau) 1_{\{\tau=\tau_\theta\}}
\]

Bottom path, the firm fails due to other creditors’ run
Derivation of Equilibrium (1)

- How to derive $V(y; y_*)$, i.e., the individual creditor’s continuation value, given $y$ and others running at $y_*$?
- HJB equation

\[
\rho V(y; y_*) = \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} + r + \phi \left[ \min (1, y) - V(y; y_*) \right] \\
+ \theta \delta 1_{\{y < y_*\}} \left[ \min (L + Ly, 1) - V(y; y_*) \right] \\
+ \delta \max_{\text{rollover or run}} \{0, 1 - V(y; y_*)\}.
\]

- Potentially we need to solve for $V(y; y'_*, y_*)$ where $y'_*$ is the individual optimal response to $y_*$. We avoid this.
- We show any (threshold) equilibrium must be symmetric, i.e., in equilibrium this individual creditor runs at $y_*$ as well.
- So what is any individual’s value if everybody runs at $y_*$, given $y$?
- We find an equilibrium if and only if
  - $V(y_*; y_*) = 1$;
  - $V(y; y_*)$, as a function of $y$, crosses 1 only once.
- Finally we show uniqueness of $y_*$ s.t. $V(y_*; y_*) = 1$.
Derivation of Equilibrium (2)

- From now, I ignore \( y_* \) in \( V(y; y_*) \). only focus on \( y \).
- So we have two regions and \( y \) can move back and forth.
  - If \( y < y_* \), higher discount rate \( \rho + \phi + (\theta + 1) \delta \), higher payoff flow

\[
0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho + \phi + (\theta + 1) \delta] V(y) \quad \text{(below)}
\]
\[
+ \phi \min(1, y) + \theta \delta \min(L + ly, 1) + r + \delta;
\]
  - If \( y \geq y_* \), lower discount rate \( \rho + \phi \), lower payoff flow

\[
0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y) + \phi \min(1, y) + r. \quad \text{(above)}
\]

- The solutions \( V(y) \) have to be pasted together at \( y_* \):

\[
V(y = y_*^+) = V(y = y_*^-) : \text{value matching}
\]
\[
V'(y = y_*^+) = V'(y = y_*^-) : \text{smooth pasting}
\]

- Note: sometimes you see smooth-pasting as optimality condition (say, option exercising; default in Leland)
  - Here, smooth-pasting has nothing to do with optimality.
  - Value matching because \( y \) is continuous at \( y_* \); smooth pasting because \( y \) can move back and forth at \( y_* \).
Derivation of Equilibrium (3)

- Fundamental eq. of (below): $\frac{1}{2}\sigma^2 x(x - 1) + \mu x - [\rho + \phi + (1 + \theta) \delta] = 0$, roots:
  
  $\gamma_1 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 [\rho + \phi + (1 + \theta) \delta]}}{\sigma^2} < 0,$
  
  $\eta_1 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 [\rho + \phi + (1 + \theta) \delta]}}{\sigma^2} > 1$

  and

  $V(y) = A_{below} + B_{below} y + C_{below} y^{-\gamma_1} + D_{below} y^{\eta_1}$

- Fundamental eq. of (above): $\frac{\sigma^2}{2} x(x - 1) + \mu x - (\rho + \phi) = 0$, roots

  $\gamma_2 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 (\rho + \phi)}}{\sigma^2} < 0,$
  
  $\eta_2 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 (\rho + \phi)}}{\sigma^2} > 1$

  and

  $V(y) = A_{above} + B_{above} y + C_{above} y^{-\gamma_2} + D_{above} y^{\eta_2}$
A and B can be determined directly by setting $C = D = 0$, plugging, matching functions (just as I calculate the fundamental value $F(y)$).

Two usual tricks to pin down $C$ & $D$.

- For below, $C_{\text{below}} = 0$ because $y^{-\gamma_1} \to \infty$ when $y \to 0$.
- For above, $D_{\text{above}} = 0$ because $y^{\eta_2} \to \infty$ when $y \to \infty$.

We need to determine $D_{\text{below}}$ and $C_{\text{above}}$. But we have value-matching and smooth-pasting. Done!

- $D_{\text{below}}$ and $C_{\text{above}}$ are solved as functions of $y_*$.
- In paper, $\min(L + ly, 1)$ and $\min(y, 1)$ create a bit more complication.

So far, standard and useful (Leland models, investment models, real options).

A great (intuitive) book: The art of smooth pasting by Dixit.
Derivation of Equilibrium (5)

- Equilibrium $y_*$ is defined as $V(y_*; y_*) = 1$.
- What is hard and requires certain technical skills is to show:
  1. It is an equilibrium: if $V(y_*; y_*) = 1$, then $V(y; y_*)$, as a function of fundamental $y$, crosses 1 only once.
  2. It is unique, i.e., there exists only ONE $y_*$ so that $W(y_*) \equiv V(y_*; y_*) = 1$.
     - We show $W(0) < 1$, $W(\infty) > 1$ and $W'(\cdot) > 0$.
- Conclusion: exists a unique equilibrium threshold $y_*$ s.t. $V(y_*; y_*) = 1$.
- Intuition:
  - Equilibrium uniquely defined in upper and lower dominance regions.
  - Knowing future maturing creditors will not run in dominance regions, backward induction uniquely determines equilibrium in the middle.
  - This is similar to global games, replacing $y$ by private signal.
The Unique Monotone Equilibrium

- Strategic uncertainty originates from time-varying fundamental.
  - e.g., Frankel and Pauzner (2000).
  - In contrast to Carlsson and van Damme (1993) and Morris and Shin (1998), strategic uncertainty arises from noise in private signals.

- Requires a well spread-out fundamental process.
  - Does not rely on specific information structure and immune from information revealed by market prices, e.g., Angeletos and Werning (2006) and Hellwig, Mukherji and Tsyvinski (2006).

- Difference from Frankel and Pauzner (2000).
  - Strategic complementarity exists in continuation values not in flow payoffs; deletion of dominated strategies not applicable.
  - We use a guess-and-verify approach.
Debt run externality

Each creditor’s run imposes an externality on the other creditors who are locked in

\[
\begin{array}{c|c|c|c}
\text{Possible firm outcomes} & \text{Choice of maturing creditors} & \text{Run} & \text{Rollover} \\
\hline
\text{Probability} & \text{failed} & \text{survived} & \text{survived} \\
\text{Payoff of current maturing creditors} & \theta \delta dt & 1 - \theta \delta dt & 1 \\
\text{Payoff of future maturing creditors} & \tilde{L}(y) & 1 & V(y) \\
\end{array}
\]
Consider an unexpected drop in liquidation recovery rate $\alpha$.

A creditor’s optimal response $y'$ to other creditors’ threshold $y_\ast$. 

- Best Response $y'(y_\ast)$ when $\alpha$ is low
- Best Response $y'(y_\ast)$ when $\alpha$ is high

Graph showing the relationship between $y$ and $y_\ast$. The graph illustrates the 45-degree line and the optimal responses $y_\ast,0$, $y_\ast,1$, and $y_\ast,2$. The graph also highlights the best responses $y'(y_\ast)$ for different values of $\alpha$. 

Diagram: 45-degree line, $y$-axis, $y_\ast$-axis, $y_\ast,0$, $y_\ast,1$, $y_\ast,2$, best response $y'(y_\ast)$.
Predicting One Year Default Probability of Merrill Lynch

- Using Merrill Lynch equity movement to estimate $y$, then predict 1-year default probability
Effects of Fundamental Volatility

- Volatility affects each creditor in three channels:
  - Insolvency risk, causing $y_*$ to increase with $\sigma$;
  - Rollover risk (strategic uncertainty), causing $y_*$ to increase with $\sigma$;
  - Embedded option, causing $y_*$ to decrease with $\sigma$. 

![Panel A: Equilibrium Threshold](image)

![Panel B: One-Year Default Probability](image)
Effects of Credit Lines

- Credit lines can temporarily sustain a firm under runs.
  - Common intuition: stronger credit lines should deter runs.
- When volatility is sufficiently large, credit lines exacerbate runs because fundamental can deteriorates during the period the firm lives on credit lines.
  - Uncertain government bailouts can be counter productive.
Effects of Debt Maturity

- Common intuition: longer debt maturities mitigate runs.
- Two offsetting effects of longer maturities:
  - 1) external: the firm faces less frequent rollover with other creditors and thus less likely to fail under runs.
  - 2) internal: longer lock-in effect for each creditor, which motivates runs, especially severe when volatility is high.
- Longer maturities exacerbate runs when volatility is sufficiently high.
  - consistent with experience of runs on ABCP, e.g., Covitz, Liang, and Suarez (2012, JF).
Further Discussion

- **Synchronous vs Asynchronous Debt Structure**
  - It is common for firms to spread out debt expirations.
  - The synchronous structure leads to more severe runs than the static-rollover benchmark when volatility is sufficiently high.
  - Which structure is optimal?

- **Optimal Debt Maturity**
  - Cheng and Milbradt (2012 RFS) extends our model to allow the firm switching b/w two projects: one with high growth and low volatility, the other with low growth and high volatility.
  - Suarez, Schroth and Taylor (2014 JFE) allows for endogenous interest rate
  - The optimal debt maturity trades off discipline on risk shifting and debt run risk.

- **Spillover and Systemic Risk**
  - When firms hold similar assets and face a downward sloping curve, runs on one firm can spill over to other firms.
  - Each firm’s optimal debt structure and debt maturity depend on its own characteristics (fundamental volatility and asset illiquidity) and peer characteristics.
Other ways to model runs

- In He-Xiong, agents have synchronous information, but are locked in asynchronously
- In Abreu-Brunnermeier and He-Manela, agents can act (i.e., withdraw) any time, but get to be informed asynchronously
  - Both settings introduce heterogeneity in a tractable way
- He-Manela interpret asynchronous timing of being informed as rumor spreading in a social network
- Two interesting things missing there
  - Rumor spreading speed is exogenous...
  - There is no public price to aggregate information (as standard critique to global games)
- Albagli, Hellwig, and Tsyvinski (2014), an alternative nonlinear setting to study information aggregation, though not dynamic