Lecture 2A: Static Global Games and Applications in Finance

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Coordination and Multiple Equilibria

- Many economic scenarios feature coordinations among economic agents
  - Bank runs a la Diamond-Dybvig
  - Currency attack a la Morris-Shin
- Basically, strategic **complementarity**, as agents want do something if others do the same thing
  - Typically in financial market trading game features strategic **substitutability**
    - If other people buy, pushing up price, so you want to sell
- Multiple equilibria emerge easily with strategic complementarity
- Global games technique help get a unique equilibrium
Carlsson and van Damme (1993): Setting

- Two players, binary actions "invest" or "not invest," fundamental $\theta$
- Normal form

<table>
<thead>
<tr>
<th></th>
<th>Invest</th>
<th>Not Invest</th>
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<tbody>
<tr>
<td>Invest</td>
<td>$\theta, \theta$</td>
<td>$\theta - 1, 0$</td>
</tr>
<tr>
<td>Not Invest</td>
<td>$0, \theta - 1$</td>
<td>$0, 0$</td>
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- Basically for agent $i$, the payoff of investing is
  $$\theta - 1 \{ j \text{ does not invest} \}$$

- Unique equilibrium if $\theta > 1$ (both invest) and $\theta < 0$ (nobody invest)
- Two equilibria if $\theta \in (0, 1)$: either both invest, or nobody invest
- Problem: common knowledge in equilibrium strategies given complete information
  - Perfect guessing each other’s strategies
- Introduce private information about $\theta$ to break it
Nobody knows $\theta$ exactly, but observe a private signal

$$x_i = \theta + \varepsilon_i, \ i = 1, 2$$

where $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ i.i.d. across agents

Bayesian updating. For simplicity, say the prior of $\theta$ is "improper prior" or noninformative prior (equally likely over real line)

- Given $x_i$, the posterior of $\theta \sim \mathcal{N}(x_i, \sigma^2)$
- Given $x_i$, the posterior of opponent's signal $x_j \sim \mathcal{N}(x_i, 2\sigma^2)$

Agent $i$’s conditional payoff

$$\mathbb{E} [\theta - 1 \{j \text{ not invest}\} | x_i] = \underbrace{x_i}_{\text{fundamental}} - \underbrace{\Pr (j \text{ not invest} | x_i)}_{\text{strategic}}$$

Two distinctive roles played by the signal $x_i$

- First term: $x_i$ tells me something about fundamental
- Second term: $x_i$ tells me something about the distribution of agent $j$’s signal $x_j$ and thus his strategy
The second term captures the idea of "guessing" each other's strategy, in a simple but powerful way.

Suppose that everybody follows a cutoff rule with threshold \( k \):

\[
\begin{cases} 
\text{Invest} & \text{if } x > k \\
\text{Not invest} & \text{otherwise}
\end{cases}
\]

Because \( x_j | x_i \sim \mathcal{N}(x_i, 2\sigma^2) \), the probability of agent \( j \) not investing is

\[
\Pr(x_j \leq k | x_i) = \Phi \left( \frac{k - x_i}{\sqrt{2\sigma}} \right)
\]

So, agent \( i \) will invest if and only if

\[
x_i - \Phi \left( \frac{k - x_i}{\sqrt{2\sigma}} \right) \geq 0
\]
Unique Equilibrium Threshold

- The equilibrium cutoff $k$: when $x_i = k$, agent $i$ indifferent between invest and not invest

$$k - \Phi \left( \frac{k - k}{\sqrt{2\sigma}} \right) = 0 \Rightarrow k = \frac{1}{2}$$

- So, the unique equilibrium is that every agent invests if his/her signal $x_i > 1/2$

- Intuition:
  - Symmetry: when receiving $x_i = k$, the probability of $j$ getting signal $x_j$ below $k$ is 0.5
  - Strategic uncertainty (guessing each other) implies the second term to be 0.5
  - The first fundamental term have to be 0.5 for the equilibrium threshold
The role of Dominance Regions

- The assumption of threshold strategy can be relaxed
- Starting from upper and lower dominance regions, unique equilibrium survives after iterated deletion of strictly dominated strategies
  1. Say $x_i \sim U(\theta - \varepsilon, \theta + \varepsilon)$. At $x_i = 1 + \varepsilon$ you know $\theta \geq 1$, invest for sure, **even if $j$ is for sure not investing**
  2. The same logic applies to $j$ so you know $j$ invests if $x_j \geq 1 + \varepsilon$
  3. Now $x_i = 1 + 0.99\varepsilon$ so $x_j$ is centered around $1 + 0.99\varepsilon$. This implies you know $j$ **invests with probability** $0.5 - \delta(\varepsilon) \sim 0.5$
    - Here $\delta(\cdot)$ is a continuous function
  4. Comparing to Step 1, you should invest at $x_i \geq 1 + 0.99\varepsilon$
  5. Symmetrically, $j$ will invest if $x_j \geq 1 + 0.99\varepsilon$
  6. How about $1 + 0.98\varepsilon$? So on so forth
- You can do the same thing starting from the lower end $\theta = -\varepsilon$, with the only difference of "not investing"
- Both sides collide at the equilibrium threshold $k = 0.5$
Discontinuity at the full information case

- Magically, the equilibrium threshold $k = 0.5$ does not depend on how noisy the private signal $\sigma$ is!
- Say $\sigma \to 0$ so that the game seems to converge to the full information case, the equilibrium is still unique
- Interestingly, it implies a negligible probability of getting inside the dominance regions
  - But iterated deletion of dominated strategies, which relies on support, still works
  - You often hear high-order beliefs....this is one example. Here, I think you think I think you think.....can get to dominant regions
- Although fundamental uncertainty shrinks, the strategic uncertainty effect remains at 0.5
  - Hence agents invest only when the fundamental is above 0.5
Public versus Private Information: Setting

- Consider a more elegant Normal framework with public information
- A continuum of agents
  - Not investing: 0; Investing: \( \theta + l - 1 \) where \( l \) is the proportion of people investing
- \( \theta \) is fundamental, with prior

\[
\theta \sim \mathcal{N} \left( y, \tau^2 \right)
\]

where \( y \) is the public signal (everybody observes it)

- Previous setting, improper prior, it is as if \( \tau^2 \rightarrow \infty \)
- Private signal

\[
x_i = \theta + \varepsilon_i \text{ with } \varepsilon_i \sim \mathcal{N} \left( 0, \sigma^2 \right)
\]

- The posterior of \( \theta \) given \( x_i \) and \( y \) is

\[
\theta \sim \mathcal{N} \left( \frac{\sigma^2 y + \tau^2 x_i}{\sigma^2 + \tau^2}, \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} \right)
\]

- Consider threshold strategy \( k \) so that investing iff \( \bar{\theta} > k \)
Strategic Uncertainty

Given $x_i$ and $y$, what is his belief about other agent’s signal $x' = \theta + \varepsilon'$?

- Obviously this gives his guessing of other agent’s actions

- The posterior of $x'$ given $x_i$ and $y$

$$x' \sim \mathcal{N}\left(\bar{\theta}, \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} + \sigma^2 = \frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}\right)$$

- The probability of other agents’ posterior mean $\bar{\theta}' = \frac{\sigma^2 y + \tau^2 x'}{\sigma^2 + \tau^2} > k$ is (plugging $x' = \bar{\theta} +\text{noise}$)

$$1 - \Phi\left(\frac{k + \frac{\sigma^2}{\tau^2} (k - y) - \bar{\theta}}{\sqrt{\frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right) = I$$

- I have used law of large numbers: it is the population of investing
Equilibrium Characterization

- At equilibrium threshold $\overline{\theta} = k$, indifference
  $\overline{\theta} + l - 1 = k + l - 1 = 0$ implies
  
  $$k = \Phi \left( \frac{k + \frac{\sigma^2}{\tau^2} (k - y) - k}{\sqrt{2\sigma^2\tau^2 + \sigma^4}} \right) = \Phi \left( \frac{\frac{\sigma^2}{\tau^2}}{\sqrt{2\sigma^2\tau^2 + \sigma^4}} (k - y) \right)$$

- Again, LHS is fundamental $k$, RHS is strategic uncertainty

- Define
  
  $$\gamma(\sigma, \tau) \equiv \frac{\sigma^2}{\tau^4} \left( \frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2} \right)$$

  then the equilibrium threshold $k$ satisfies
  
  $$k = \Phi \left( \sqrt{\gamma} (k - y) \right)$$

- The only difference from previous example: introduce $\tau$ (precision of public signal) and $\sigma$ (precision of private signal)
  
  - Public signal gives some prior. When $\tau \to \infty$ (improper prior), $\gamma \to 0$, so $k = \Phi \left( 0 \cdot (k - y) \right) = 0.5$ always
Multiplicity of Equilibria (1)

- With public signal, we might get back to multiple equilibria
- Key equation

\[ v(k) = k - \Phi(\sqrt{\gamma}(k - y)) = 0 \]

- Obviously \( v(\pm\infty) = \pm\infty \) so there must exist solution(s)
- Whether the solution is unique depends on whether \( \gamma \leq 2\pi \)
- Say public signal \( y = 0.5 \). Then \( k = 0.5 \) is an equilibrium. Its derivative at that point is

\[ v'(k = 0.5) = 1 - \sqrt{\gamma}\phi(\sqrt{\gamma}(0.5 - 0.5)) = 1 - \sqrt{\gamma}\frac{1}{\sqrt{2\pi}} \]

so if \( v'(k = 0.5) < 0 \) then there must be at least three equilibria
Multiplicity of Equilibria (2)
The equilibrium satisfies $y = k - \frac{1}{\sqrt{\gamma}} \Phi^{-1} (k)$.
The Publicity Multiplier

- The public signal $y$, as observed by everybody, carry more weight in coordination and thus the equilibrium
- Recall posterior fundamental $\bar{\theta} = \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2}$
- How much a player’s private signal must adjust to compensate for a given change in public signal, so that he is indifferent between investing and not?
- Without strategic uncertainty effect, $\bar{\theta} = \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} = k$ so
  \[ \frac{dx}{dy} = -\frac{\sigma^2}{\tau^2} \]
- With strategic effect, $\frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} = \Phi \left( \sqrt{\gamma} \left( \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} - y \right) \right)$ so
  \[ \frac{dx}{dy} = -\frac{\sigma^2}{\tau^2} + \sqrt{\gamma} \phi (\cdot) \]
- The smaller the $\tau$ (more precise public signal), the greater the $\gamma$, the larger the publicity multiplier
- When the publicity multiplier is too large, the multiplicity of equilibria comes back!
The Publicity Multiplier: Implications

- Public signals play a role in coordinating outcomes beyond its mere information content
  - Financial market "overreact" to announcement for Fed. Could be rational
  - Wall street journals or Washington post can affect the agent’s belief on others

- Interesting empirical paper: Chwe (1998) on Superbowl advertisement
  - The price of advertisement increases more than linearly in the number of viewers
  - The premium may reflect "coordination value" of consumers’ purchase decisions
  - Indeed, the premium mainly concentrate on coordination goods like "Apple Macintosh" or "Beer," but not in "Batteries"
A direct application of global games technique in Diamond-Dybvig

Good application: key insight is the same, but need to deal with realistic complications

Why? In bank runs setting, naturally strategic complementarity does not hold globally
  - Rochet-Vives (2004) assume this away...

Say liquidation value $l = 0$ and default occurs for sure. Strategic complementarity always holds
  - Say $n$ other agents run. My incentives of running is constant (so weakly increasing in $n$):
    \[
    V(run|n) - V(no\_run|n) = 0 - 0 = 0
    \]

If liquidation value $l > 0$ which is Diamond-Dybvig framework ($l = 1$)...
  - Then $V(run|n) - V(no\_run|n) = 1/n$ is decreasing in $n$
  - The more the other people are running, the less the incentives for me to run with them (as all of us have to share the liquidation value)
Key derivations (1)

- Utility $u(\cdot)$, prob $\lambda$ being early, LT project returns $R$ with prob $\theta$
- Agents observe $\theta_i = \theta + \epsilon_i$, $\epsilon_i \sim U[-\epsilon, \epsilon]$; $r_1$ early payment
- Early type withdraw early; say equilibrium running threshold for late type is $\theta^*$
- Given true fundamental $\theta$ and $n$ agents withdrawing early, late type calculates the utility differential $v(\theta, n)$ between stay and run,

$$v(\theta, n) = \begin{cases} 
\theta u \left( \frac{1-nr_1}{1-n} R \right) - u(r_1) & \text{if } \lambda \leq n \leq \frac{1}{r_1} \\
0 - \frac{1}{nr_1} u(r_1) & \text{if } \frac{1}{r_1} \leq n \leq 1
\end{cases}$$

- If not enough resource to serve early withdrawals, random rationing
- Only one-sided strategic complementarity
  - Strategic complementarity holds for $n \in \left[ \lambda, \frac{1}{r_1} \right]$
  - Strategic complementarity fails for $n \in \left[ \frac{1}{r_1}, 1 \right]$
Key derivations (2)

- Agent $\theta_i$ is calculating expected utility differential
  \[
  \Delta (\theta_i, \theta^*) = \frac{1}{2\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} \nu (\theta, n (\theta, \theta^*)) \, d\theta
  \]

- Improper prior (agnostic prior) + $\theta_i = \theta + \varepsilon_i$, $\varepsilon_i \sim U [-\varepsilon, \varepsilon] \Rightarrow$
  Posterior of $\theta \sim U [\theta_i - \varepsilon, \theta_i + \varepsilon]$

- $n (\theta, \theta^*)$: expected \# of withdrawals = early + (late with $\theta_i' < \theta^*$), given true fundamental $\theta$
  \[
  n (\theta, \theta^*) = \begin{cases} 
  1 & \text{if } \theta < \theta^* - \varepsilon \\
  \lambda + (1 - \lambda) \frac{\theta^* + \varepsilon - \theta}{2\varepsilon} & \text{if } \theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon] \\
  \lambda & \text{if } \theta > \theta^* + \varepsilon
  \end{cases}
  \]

- Equilibrium condition: $\Delta (\theta_i = \theta^*, \theta^*) = 0$
Results and Drawbacks

- They prove the uniqueness of equilibrium, without assuming ad hoc restrictions on strategy space (say, threshold strategy)
  - It is a big deal for this paper

- Diamond-Dybvig finds an optimal banking solution for liquidity insurance. But it is subject to bank runs, and without knowing the probability of bank runs, we do not know what is the optimal solution ex ante
  - You can put sunspot probabilities to Diamond-Dybvig... but arbitrary

- In Goldstein-Pauzner, endogenous $\theta^*$ depends on $r_1 \Rightarrow$ the running probability is a function of solution itself (the payment $r_1$) $\Rightarrow$ optimal demand-deposit contract

- One unsatisfactory part: how do we interpret the upper dominating threshold?
  - When $\theta$ is high enough, even if everyone is running, I want to stay—maybe ok for just one bank
  - If everyone is running on the financial system, probably everything is destroyed
Concluding Remarks

- Critique: global games rely on delicate information structure
  - There is some unobservable fundamental, and everybody gets a private signal about it (and guessing each other)
  - How about financial market which aggregates all the private information?
    - Angelatos-Werning: Grossman-Stiglitz setting, every thing falls apart!

- To its heart, global games work because it breaks "common knowledge"

- Private signal introduces heterogeneity to individual agents, and that goes a long way to pin down a unique equilibrium

- Is there another simple way to introduce heterogeneity without introducing private signals?

- He-Xiong Dynamic Debt Runs (RFS, 12), unique equilibrium in a run-like model
  - Time-varying fundamentals,
  - Debt holders are taking rollover actions at different times
  - Because fundamentals are changing at different times, you get heterogeneity!