Indeterminacy and Sunspots in Macroeconomics

Wednesday September 6th: Lecture 5

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Topics for Lecture 5

• Sunspots (Cass-Shell paper)
• Incomplete participation
• Incomplete markets
• The Farmer-Woodford Model and self-fulfilling prophecies
Reading


• Farmer *Macroeconomics of Self-Fulfilling Prophecies* MIT Press Chapter 10
Sunspots: the Cass-Shell paper

• **Intrinsic uncertainty**: affects preferences endowment or technology

• **Extrinsic uncertainty**: does not affect any of the fundamentals
Sunspots: the Cass-Shell paper

- **Question**: Can there be an equilibrium in an economy with complete financial markets where allocations are different across states that differ only in extrinsic uncertainty?
- **Answer**: Yes. If there is incomplete participation in financial markets
- This is **ALWAYS** true in an overlapping generations model
Sunspots: the Cass-Shell paper

“Do sunspots matter?” *JPE* 1983 9(2)

- Two generations
- Complete markets
- Two dates
- Two commodities at date 2
- Two states at date 2, $\alpha$ and $\beta$
- No commodities at date 1
Two Market Structures

- Cass and Shell consider two market structures:
  
  (A) Securities Markets
  - As in Arrow’s formulation of GE

  (B) Contingent Commodities
  - As in Debreu’s formulation of GE
The Timing of Events

Timing

$G_1$ born

Securities traded

Sunspot

$G_2$ born

Commodities traded
(A) Arrow Securities

1) The securities market opens. Agents of generation 1 trade securities contingent on realization of sunspot.
2) A sunspot is realized
3) Generation 2 is born
4) Trade takes place in the commodities markets
(B) Contingent Commodities

1) Market opens at the beginning of time. Agents trade the following commodities,

\[ \{x^1(\alpha), x^2(\alpha), x^1(\beta), x^2(\beta)\} \]

2) In this structure Generation 2 is present, but is not allowed to trade across states of nature
Securities and Arrow Securities

Suppose that there are 2 periods, $\ell$ goods and $n$ states in period 2. Suppose further that there are $m$ securities and the security $a^i$ pays a vector $b = (b_1^i, b_2^i, \ldots b_n^i)$ of dollars in each state $s = \{1, \ldots, n\}$

For example, the payoff of the $j'$th Arrow security

$$b^j = (0, \ldots, 1, \ldots 0)_{1,\ldots,j,\ldots,n}$$

pays one dollar in state $j$ and 0 otherwise
Securities and Arrow Securities

If the matrix

\[
B = \begin{bmatrix}
  b_1^1 & \cdots & b_1^n \\
  \vdots & \ddots & \vdots \\
  b_m^1 & \cdots & b_m^n
\end{bmatrix}
\]

Has full row rank, we say that markets are complete

A complete set of Arrow securities is the special case when

\[
B = \begin{bmatrix}
  1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 1
\end{bmatrix} \equiv I_n
\]
Securities and Arrow Securities

When markets are complete, the household faces a single budget constraint and can transfer income between the present and all future states.

When markets are incomplete, the household faces multiple budget constraints (Can you explain why?)
Incomplete Markets and Incomplete Participation

Equilibria are Pareto Inefficient when markets are incomplete

Some economists argue that markets are obviously incomplete because we do not observe securities designed to be contingent on every observable contingency

Other economists argue that this doesn't matter. If a market is important: it will be created
Incomplete Markets and Incomplete Participation

Although it seems unlikely (to me) that markets are complete: Cass and Shell assume complete markets in order to make a point.

Even if every conceivable security is traded in financial markets: those markets will still lead to Pareto Inefficient outcomes if there is incomplete participation
Incomplete Participation

Is incomplete participation a reasonable assumption?

**YES:** because we cannot trade securities in markets that open before we are born
An Equivalence Result

• If
  • there are complete markets
  • preferences are time consistent (Von-Neumann Morgenstern preferences satisfy this assumption)

• Then
  • Representation (A) with complete financial markets is equivalent to
  • Representation (B) with complete contingent commodities
Sunspots

• Since (A) and (B) are equivalent, Cass and Shell proceed with the complete contingent commodity representation
Some Definitions

\[ x_h(\alpha) \equiv \{x_h^1(\alpha), x_h^2(\alpha)\} \quad \text{vectors of contingent commodities} \]

\[ x_h(\beta) \equiv \{x_h^1(\beta), x_h^2(\beta)\} \]

\[ h \quad \text{indexes households} \]

\[ \alpha \text{ and } \beta \quad \text{index the state} \]

Superscripts 1 and 2 \quad \text{index commodities}
The problem of $G_1$

$$\max U \equiv \sum_{s=\alpha, \beta} \pi_s u_h [x_h(s)]$$

$$p \cdot x_h \leq p \cdot \omega_h$$

where

$$x_h \equiv \{x_1^1(\alpha), x_2^2(\alpha), x_1^1(\beta), x_2^2(\beta)\}$$

and

$$p \equiv \{p^1(\alpha), p^2(\alpha), p^1(\beta), p^2(\beta)\}$$
The two problems of $G_2$

$$\max U \equiv u_h[x_h(s)], \quad h \in G_2, \quad s = \alpha, \beta$$

$$p(s) \cdot x_h(s) \leq p(s) \cdot \omega_h(s)$$

where

$$x_h(s) \equiv \{x^1_h(s), x^2_h(s)\}$$

and

$$p(s) \equiv \{p^1(s), p^2(s)\}$$
Types of Uncertainty

Definition 1:
Uncertainty is intrinsic if it affects preferences or endowments
(e.g. the weather, taste shocks)

Uncertainty is extrinsic if it does not affect preferences or endowments
(e.g. opinion piece in the Wall Street Journal with no basis in fact. Fake news?)
An Assumption

Assumption 1: All uncertainty is extrinsic

This implies that

\[ \omega_h(\alpha) = \omega_h(\beta) \quad \text{for} \quad h \in G_1 \cup G_2 \]
A Definition (Sunspots)

Definition 2: Sunspots matter if there is an equilibrium where

\[ x_h(\alpha) \neq x_h(\beta) \]

for at least one \( h \)
Two Definitions of Pareto Efficiency

If there are two possible future states: state $\alpha$ and state $\beta$

Is the person born tomorrow in state $\alpha$ the same person as if she were born in state $\beta$?

Why is that an interesting question?
Ex Post Pareto Efficiency

If we consider Mr. $\alpha$ to be a different person from Mr. $\beta$ then sunspot allocations are Pareto Efficient. Mr. $\alpha$ is treated differently from Mr. $\beta$ but that’s ok because they are different people.

We say that a sunspot equilibrium, in this case, is Ex Post Pareto Efficient
Rawls and the Veil of Ignorance

Imagine putting yourself in a state, before you are born. The Philosopher John Rawls calls this experiment, the veil of ignorance.

Behind the veil of ignorance, you face the following gamble. With probability $p$ you can be born as a rich person in Switzerland in 2017 or with probability $1 - p$ you can be born as a slave in Sparta in 1BC.

Alternatively, you can avoid this gamble and be born as a middle class citizen of a European country in 1950.
Many of us would pay quite a lot to avoid a probability of the slavery option.

Under the veil of ignorance, we consider Mr. $\alpha$ to be the same person as Mr. $\beta$. We say that a sunspot equilibrium in this case is **Ex Ante Pareto Inefficient** because it randomizes over the good and the bad outcomes even when the mean gamble is feasible.
The Sunspot Propositions:

Proposition 1:
If $G_2 \equiv \emptyset$ (unrestricted participation) then sunspots do not matter

Proposition 2:
Sunspot equilibria are Ex Ante Pareto Inefficient

Proposition 3:
If $G_2 \neq \emptyset$ there may be equilibria where sunspots matter
Sunspots and Indeterminacy

The proof of the existence of sunspots is constructive

Cass and Shell set up a model with three equilibria and they construct a new equilibrium that randomizes over two of the equilibria in the world with certainty

That would not be possible if there was complete participation. *(Can you explain why?)*
Sunspots and Indeterminacy

We have studied models with dynamic indeterminacy. And we have learned about sunspots. Next: I will show how to put the two together

Sunspot equilibria are constructed by randomizing over the equilibria of models with multiple certainty equilibria

It is easy to construct sunspot equilibria in models with an indeterminate set of equilibria because there are so many equilibria to randomize over
The Farmer-Woodford Example

The FW example is a two-period overlapping generations model with production

\[ y_t \quad \text{Output} \quad c_t \quad \text{Consumption} \]

\[ n_t \quad \text{Labor supply} \quad M_t \quad \text{Money} \]

\[ p_t \quad \$ \text{ Price of a good} \quad g \quad \text{Govt. Spending} \]
Households

\[
\max U = -n_t^2 + E_t [c_{t+1}]
\]

utility function

\[y_t = n_t\]
technology

\[p_t n_t = M_t\]
first period budget constraint

\[p_{t+1} c_{t+1} = M_t\]
second period budget constraint

\[p_{t+1} c_{t+1} = p_t n_t\]
life-cycle budget constraint
Solution

\[ \max_{n_t} U = -n_t^2 + E_t \left[ \frac{p_t}{p_{t+1}} n_t \right] \]

\[ n_t = E_t \left[ \frac{p_t}{p_{t+1}} \right] \quad \text{Labor supply} \]

\[ y_t = E_t \left[ \frac{p_t}{p_{t+1}} \right] \quad \text{Output supply} \]
Money Demand

\[ n_t = y_t = \frac{M_t}{p_t} \equiv m_t \]

\[ m_t = E_t \left[ \frac{p_t}{p_{t+1}} \right] \]

Output equals labor supply equals real value of money balances
Government

\[ M_t - M_{t-1} = p_t g \]  Government prints enough money to purchase \( g \) goods

\[ m_t - m_{t-1} \frac{p_{t-1}}{p_t} = g \]

\[
\frac{p_{t-1}}{p_t} = \frac{m_t - g}{m_{t-1}}
\]
Market Clearing

\[
\frac{\frac{p_t}{p_{t+1}}}{m_t} = \frac{m_{t+1} - g}{m_t}
\]

Endogenous money supply

\[
m_t = E_t \left[ \frac{p_t}{p_{t+1}} \right]
\]

Household optimization
Equilibrium

\[ \frac{p_t}{p_{t+1}} = \frac{m_{t+1} - g}{m_t} \]

Endogenous money supply

\[ m_t = E_t \left[ \frac{p_t}{p_{t+1}} \right] \]

Household optimization

\[ m_t = E_t \left[ \frac{m_{t+1} - g}{m_t} \right] \]

Equilibrium
Any stochastic process that satisfies this equation is an equilibrium

\[ m_t = E_t \left[ \frac{m_{t+1} - g}{m_t} \right] \]

\[ m_1 = \frac{M_1}{p_1} \]

\( M_1 \) is the initial dollar value of the money supply. There is no economic condition pinning down \( p_1 \)
Non-stochastic Equilibria

\[ m_{t+1} = m_t^2 + g \]

Non-stationary non-stochastic equilibria

\[ m^2 - m + g = 0 \]

Stationary non-stochastic equilibria

\[ \bar{m} = \frac{-1 \pm \sqrt{1 - 4g}}{2} \]
Characterizing Equilibria

Two stationary equilibria

\[ m_{t+1} = m_t^2 + g \]

\[ \bar{m}^1 = \frac{-1 - \sqrt{1 - 4g}}{2} \]

\[ \bar{m}^2 = \frac{-1 + \sqrt{1 - 4g}}{2} \]
Characterizing Equilibria

\[ m_{t+1} = m_t^2 + g \]

A non-stationary equilibrium

45° line
Let’s recap

I have provided a simple example of a rational expectations model with no fundamental uncertainty where there are two stationary equilibria and there is a continuum of non-stationary equilibria.

Check for yourself that and $m_1 \in [\bar{m}^1, \bar{m}^2)$ is the first element of an equilibrium sequence.

All of these sequences converge, asymptotically, to $\bar{m}^1$. 
Sunspots and Indeterminacy

There is also an equilibrium in which

\[ \{ m_t \}_{t=1}^{\infty} = \{ m^2 \}_{t=1}^{\infty} \]

Which equilibrium will occur?

The answer depends on the initial price level
Sunspots and Indeterminacy

If

\[ p_1 = \frac{M_1}{m^2} \]

the economy will begin at, and remain at, the upper steady state forever. If instead,

\[ p_1 = \frac{M_1}{m^1} \]

the economy will begin at, and remain at, the lower steady state forever.
Sunspots and Indeterminacy

If

\[ p_1 \in \left[ \frac{M_1}{m^1}, \frac{M_1}{m^2} \right], \]

There will be a self-fulfilling sequence of beliefs about future prices that will cause real balances to converge to the lower steady state.

All of these sequences are Pareto Inefficient.
Sunspots and Indeterminacy

We can also add a random variable (a sunspot) to the difference equation that describes equilibria

\[ m_{t+1} = m_t^2 + g + \varepsilon_{t+1} \]

And as long as

\[ E_t[\varepsilon_t] = 0 \]

Sequences generated by the equation obey all of the conditions that define an equilibrium
Stochastic Equilibria

\[ m_{t+1} = m_t^2 + g + \varepsilon_{t+1} \]
Summary

• Sunspot equilibria occur if allocations differ in the face of extrinsic uncertainty
• They are constructed as randomizations across the multiple equilibria of a non-stochastic model
• Monetary models always have indeterminate sets of multiple equilibria and are therefore good candidates to generate sunspot equilibria