Indeterminacy and Sunspots in Macroeconomics

Tuesday September 5\textsuperscript{th}: Lecture 4

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Topics for Lecture 4

• Choice Under Uncertainty
• Von-Neumann Morgenstern Preferences
• Risk and Intertemporal Substitution
• Contingent commodities vs Arrow securities
Reading

• Farmer, *The Macroeconomics of Self-Fulfilling Prophecies* [Chapter 9]
Time, States of Nature and Geography

In finite GE theory utility is defined over the consumption set, $X$

Households max a utility function, $u_i(x): X \equiv R_+^\ell \rightarrow R$

Debreu pointed that an element of $x$ can be indexed by date, state of nature and location.

For example, an apple is not just an apple. It might be, for example, an apple in New York city on July 5th 2025 if it rains.
Example

Two dates; 1 and 2.

Two states of nature; \( \alpha \) and \( \beta \)

\( \ell = 2 \) commodities at each date, \( x \) and \( y \)

In this example utility is defined over six objects

\[
u_i = u_i(x_1, y_1, x_2[\alpha], x_2[\beta], y_2[\alpha], y_2[\beta])\]
Dates and States

Period 1  Period 2  Period 3  Period 4

\(\alpha, \alpha, \alpha\)  
\(\alpha, \alpha, \beta\)  
\(\alpha, \beta, \alpha\)  
\(\alpha, \beta, \beta\)  

\(\beta, \alpha, \alpha\)  
\(\beta, \alpha, \beta\)  
\(\beta, \beta, \alpha\)  
\(\beta, \beta, \beta\)  

\(\beta, \beta, \alpha\)  
\(\beta, \beta, \beta\)
Putting Structure on Utility

Von-Neumann and Morgenstern, in *Theory of Games and Economic Behavior*, asked how consumers would choose gambles over a scalar; money income

Their axioms were later extended to choice over vectors.
Lotteries

A lottery, $L$, is a random variable defined over the consumption set $X$.

For example, $L$ might be a gamble in which you receive $x_1$ with probability $p$ and $x_2$ with probability $1 - p$ where $x_1$ and $x_2$ are elements of $X$. 
The VNM Axioms:

Axiom 1: Completeness

For any two lotteries $L$ and $M$

Either, $L > M$ Or, $L > M$ Or, $L \sim M$

Where $>$ means “is preferred to”
And $\sim$ means “is equivalent to “
Also $\succeq$ means “is at least as good as”
The VNM Axioms:

Axiom 2: Transitivity

For any three lotteries $L, M$ and $N$

If, $L \succeq M$ and $M \succeq N$ then $L \succeq N$
The VNM Axioms:

Axiom 3: Continuity

For any three lotteries $L, M$ and $N$

If $L \succeq M \succeq N$ then there is a real number $p \in (0,1)$ such that $pL + (1 - p)N \sim M$
The VNM Axioms:

Axiom 4: Independence of Irrelevant Alternatives

For any three lotteries $L, M$ and $N$

If, $L < M$ then for any $M$ and any $p \in (0,1]$

$$pL + (1 - p)N < pM + (1 - p)N$$
The VNM Axioms:

Axiom 5: Reduction of Compound Lotteries

For any $Z, W$ and any $p, q, r \in (0,1]$ such that $rq = p$ and any lottery

$$X = qZ + (1 - q)W$$

$$pZ + (1 - p)W \sim rX + (1 - r)W$$
The Expected Utility Theorem

Theorem: (Expected Utility)

If axioms 1-4 hold, there exists a function \( u \) that assigns a real number \( u(x) \) to all \( x \in X \) such that a lottery \( L \equiv px_1 + (1 - p)x_2 \) is preferred to a lottery \( M \equiv qx_1 + (1 - q)x_2 \) if and only if \( E[u(L)] > E[u(M)] \).

Here

\[
E[u(L)] \equiv pu(x_1) + (1 - p)u(x_2)
\]
Do These Axioms Make Sense?

Maybe: but there are many observed violations in experiments.

Importantly: even if the axioms make sense for gambles over scalars, they may not make sense for gambles over vectors.

If we adopt Debreu’s definition of utility, no violation of rationality can ever be observed: Can you explain why?
Expected Utility in Macroeconomics

The VNM axioms were formulated for gambles of a scalar: ‘money’

Macroeconomists define expected utility over an infinite sequence.

Define $C^t \equiv \{C_1, C_2, C_3, ..., C_t\}$

Then macroeconomists assume that there exists a representative person who maximizes the expected value of a VNM utility function

\[ E[U] \equiv E_1[u(C^t)] \]
Expected Utility in Macroeconomics

Suppose that uncertainty lives on a tree

Define

\[ S^t = \{ S_1, S_2, S_3, ... S_t \} \]

\( S^t \) is called a *history* It is a random sequence.

If there are two possible events every period then \( S_1 \) and \( S^1 \) have one element, \( S_2 \) and \( S^2 \) have two elements, \( S_3 \) and \( S^3 \) have four elements, an so on.

\( S_3 \) is the set of things that can happen at date 3
\( S^3 \) is the set of paths by which am element of \( S_3 \) can be arrived at. It is a branch of the tree.
Expected Utility in Macroeconomics

Every realization of consumption at date \( t \) is a function that maps from the sequence \( S^t \) to a realization of \( C_t \)

\[
U \equiv u(C_1, C_2(S^2), C_3(S^3), ... C_t(S^t))
\]

Here,

\[
u(C_1, C_2(S^2), C_3(S^3), ... C_t(S^t))
\]

is a Von-Neuman Morgenstern utility function

When people have infinite horizons, this becomes

\[
U^\infty = E_1\{u(C_1, C_2(S^2), C_3(S^3), ... )}\}
\]
Expected Utility in Macroeconomics

Under expected utility theory, utility is linear in probabilities

\[ U^t = E_1 \{ u(C_1, C_2(S^2), C_3(S^3), \ldots C_t(S^t)) \} \]

\[ = \sum_{i \in N} \pi_i u \left( C_1, C_2(S_i^2), C_3(S_i^3), \ldots, C_t(S_i^t) \right) \]
Expected Utility in Macroeconomics

Macroeconomists also typically assume that $U$ is time separable and that people discount the future with a constant discount factor $\beta \in [0,1]$

$$U = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \nu(C_t) \right\}$$

These are very strong assumptions
Risk Aversion and Intertemporal Substitution

Risk aversion is the amount you would be willing to pay to receive the mean of a lottery as opposed to the lottery itself. It measures aversion to changes in consumption across states.

When preferences are state and time separable, risk aversion is measured by the curvature of \( v(x) \)

\[
\rho(x) = \frac{xv''(x)}{v'(x)}
\]

is called the Arrow-Pratt measure of relative risk aversion.
Risk aversion and intertemporal substitution

Intertemporal substitution is the change in your intertemporal consumption bundle for a given change in the interest rate.

It measures aversion to changes in consumption over time.

When preferences are state and time separable, the intertemporal elasticity of substitution is the inverse of the Arrow Pratt measure of risk aversion.
Intertemporal Substitution (ITS)

An optimizing consumer (with no uncertainty) would choose

\[
\frac{v'(C_{t+1})}{v'(C_t)} = \beta R_t
\]

Where \( R_t \) is the real interest factor between dates \( t \) and \( t + 1 \). Holding constant \( C_t \) the ITS, \( \eta \) is defined as

\[
\eta(x) \equiv \frac{d \log(x)}{d \log(R_t)} \equiv \frac{d \log(x)}{d \log(v'(x))} \equiv \frac{v''(x)}{x v'(x)}
\]
Risk Aversion and Intertemporal Substitution (ITS)

In the widely used parametrization where

\[ v(x) \equiv \frac{x^{1-\sigma}}{1 - \sigma} \]

The coefficient of relative risk aversion is a constant

\[ \rho(x) = \sigma \]

and the ITS is its reciprocal

\[ \eta(x) = 1/\sigma \]
Rationality as an Organizing Principle

If we take the strong view that a commodity is date, state and location dependent EVERY GOOD IS DIFFERENT

Taken literally, this means that experiments are not possible because they cannot be repeated

But although rationality defined as Debreu Chapter 7 cannot be refuted, rationality defined as Von-Neuman Morgenstern preferences can be, and has been, experimentally refuted. This is the subject of behavioral economics
Separability and Time Consistency

Time and state separability are different assumptions.

They are commonly used because they imply a separate property: time consistency.

Suppose that at date $t$ a person prefers $C^{t+s}(a)$ to $C^{t+s}(b)$. Time consistency says that at every intermediate date $t + \tau$ where $\tau < s$ the choice $C^{t+s}(a)$ is still preferred to $C^{t+s}(b)$.
The most general class of preferences that is time consistent can be written recursively as follows

\[ U_t = W(C_t, E_t[U_{t+1}]) \]

If \( W \) is affine

\[ W(x, y) = x + \beta y \]

We recover VNM preferences
Expected Utility and Recursive Utility

Additively separable VNM utility can be written recursively as

$$U_t(c^t) = v(c_t) + \beta E_t[U_{t+1}(c^{t+1})]$$

This is the expected utility representation of preferences
Epstein-Zin Preferences

A special case of recursive utility was studied by Larry Epstein and Stan Zin

\[ U_t = \left[ (1 - \beta) c_t^{1 - \frac{1}{\eta}} + \beta (E_t U_{t+1}^{1 - \rho})^{1 - \rho} \right]^{\frac{1}{1 - \frac{1}{\eta}}} \]

This function is widely used in finance because it is able to separate inter-temporal substitution (measured by \( \eta \)) from risk aversion (measured by \( \rho \)).
Representative Agents and Overlapping Generations

Time separability and state separability are not the only strong assumptions

A second strong assumption is that the world consists of a finite number of families each of whom maximizes utility over a finite horizon

I will refer to that as the representative agent assumption
Representative Agents and Economic Welfare

In economies with a finite number of infinitely lived families, the first and second welfare theorems hold.

In economies with an infinite number of finitely lived families, the first and second welfare theorems break down.

For a proof and discussion of these results: see Kehoe and Levine, *Econometrica* 1985.
Overlapping Generations and Economic Welfare

In overlapping generations economies the welfare theorems break down for two reasons:

Equilibria may be dynamically inefficient. We met this possibility in Lecture 3.

There may also exist equilibria in which non-fundamental shocks influence allocations. Cass and Shell refer to these as sunspot equilibria. We will investigate this possibility in Lecture 5.
What is a Market?

If we take Debreu Chapter 7 as our benchmark; there is a single market that opens at the beginning of time

A finite number of infinitely lived families trade an infinite set of contingent commodities

Then the world begins and the trades are realized
What is a Market?

This is the view that Ed Prescott and Robert Lucas asked us to accept when they introduced infinite horizon, complete-market economies, with a representative consumer.

Many people (myself included) think that this is not a very useful way to think about macroeconomics. In short: representative agent macroeconomics is nonsense. Read my blog post: Real Business Cycle Theory and the High School Olympics
John Hicks and Temporary Equilibrium Theory

In *Value and Capital*, Hicks formulated a different vision: Time proceeds in a sequence of weeks.

Each week people come to market to trade goods. They bring capital and financial obligations from the past.

They form beliefs about what will happen in the future.
Ken Arrow and Financial Markets

In a *Review of Economic Studies* article, *The Role of Securities in the Optimal Allocation of Risk Bearing*, (1964) Ken Arrow showed how to implement an equilibrium sequentially.
Consider the following problem of a household

\[
\max_{\{x, x'(s')\}} \quad U \equiv \sum_{i=1}^{n} \pi_i u_i(x, x'(s'_i))
\]

s.t. \( p \cdot x + \sum_{i=1}^{n} \tilde{p}(s'_i) \cdot x'(s'_i) \leq m \)

\( m \) is money income and \( \tilde{p}(s'_i) \) is a vector of present value prices
Notice that we can write the utility function as

\[ U(x, x'(s'_1), x'(s'_2), \ldots, x'(s'_n)) \equiv \sum_{i=1}^{n} \pi_i u_i(x, x'(s'_i)) \]

If the goods \( x'(s') \) are date and state dependent, the existence of equilibrium is a special case of Lecture 2. In period 1, people trade current goods, these are elements of \( x \), and state contingent future goods, these are elements of \( x'(s'_i) \) for \( i = 1, 2 \ldots n \).
Gerard Debreu and Contingent Commodity Equilibrium

Under the interpretation of Debreu Chapter 7 there is a single market that occurs in period 1. People choose \( \{x, x'(s'_1), \ldots, x'(s'_n)\} \) to solve

\[
\begin{align*}
\max \ U(x, x'(s'_1), x'(s'_2), \ldots, x'(s'_n)) \\
\text{s.t.} \quad p \cdot x + \sum_{i=1}^{n} \tilde{p}(s'_i) \cdot x'(s'_i) \leq m
\end{align*}
\]

The equilibrium is called a contingent commodity equilibrium
Arrow asked: what happens if people trade sequentially? Once in period 1 and once in period 2. He defined a security [now called an Arrow security] to be a promise to pay $1 in state $s_i$ if and only if state $i$ occurs.

If there are $n$ Arrow securities, we say that markets are complete.
Ken Arrow and Sequential Market Equilibrium

People solve the following sequential problem

\[
\max \sum_{i=1}^{n} \pi_i u_i(x, x'(s_i'))
\]

\[
\text{s.t. } p \cdot x + \sum_{i=1}^{n} Q_i a_i \leq m
\]

Here, \(a_i\) is a promise to pay \(a_i\) dollars in period 2. It can be positive or negative.

If \(a_i\) is negative we say that the consumer has shorted security \(i\)

\[
p(s_i') \cdot x'(s_i') \leq a_i,
\]

\(i = \{1, 2 \ldots n\}\)

The \(p(s_i')\) are spot prices in period 2 state \(i\)
The consumer faces a sequence of constraints

\begin{align}
\text{(1)} & \quad p \cdot x + \sum_{i=1}^{n} Q_i a_i \leq m \\
\text{(2)} & \quad p(s_i') \cdot x'(s_i') \leq a_i, \quad i \in \{1, 2 \ldots, n\},
\end{align}

By substituting for $a_i$ in (1) from (2) we get

\[ p \cdot x + \sum_{i=1}^{n} Q_i p(s_i') \cdot x'(s_i') \leq m \]

$Q_i$ is a scalar and $p(s_i')$ is a vector.
Equivalence of Contingent Commodities and Arrow Securities

Notice that under complete markets the household faces a single budget constraint and when

\[ \tilde{\rho}(s'_i) \equiv Q_i \rho(s'_i) \]

The budget constraints are the same for the two problems.
Ken Arrow and Sequential Market Equilibrium

In Arrow’s formulation of markets (following Hicks) there is a sequence of trades. At each date, people trade goods and securities.

Theorem: If preferences are time consistent, and if markets are complete, the set of equilibria under sequential markets is the same as the set of equilibria under contingent commodity trades.
Summary

• General equilibrium theory is a powerful tool
• The idea of an equilibrium is very general
  • It summarizes an important social idea
  • People make their own decisions
  • They are constrained by society
  • Society is itself the sum of those decisions