Indeterminacy and Sunspots in Macroeconomics

Tuesday September 5th: Lecture 3
Gerzensee, September 2017
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Topics for Lecture 3

• The overlapping generations model
• Dynamic inefficiency
• Failure of the welfare theorems
Reading

• Farmer, *The Macroeconomics of Self-Fulfilling Prophecies* Chapter 6

• David Gale “Pure exchange equilibrium of dynamic economic models” *JET* 1973
The Overlapping Generations Model

Simplest version
  One good
  Two period lives
  Infinite time
  One agent per generation
A Story

In the beginning of time there is a single woman: Eve

Eve is lives for one periods. Then she dies.

Eve has no means of support. She has no worldly possessions.

But she has a daughter: Evelet
A Story

Evelet will live for two periods. She will, in time, have her own daughter, Eveletta: but I am getting ahead of the story.

Evelet is endowed, in the first period of the world, with a cookie.

Eve and Evelet would both like to eat the cookie. But Evelet is selfish and she refuses to share.
Because Evelet is selfish, she eats the whole cookie and lets her mother starve.

In the second period of the world, Eveletta is born. Eveletta has a cookie in period 2 but Evelet does not.

Evelet begs Eveletta to share her cookie but Eveletta, recognizing the cruelty that Evelet inflicted on her grandmother Eve; refuses.
Evelet, like her mother Eve, starves to death.

This is a society where the old are not treated well. Can they do better?

One day, it dawns on the great-great-great-great granddaughter of Eve, her name is Cristabel, that there might be a way to avoid certain death in her golden years.
A Story

Like all of her ancestors, Cristabel eats her cookie. But, being of above average intelligence, she holds onto the cookie wrapper.

Society has invented money!
The OLG Model

Terminology ‘Samuelson’ is from David Gale
The OLG Model

Time

\[ \begin{array}{c|c|c|c}
   & G_0 & G_1 & G_2 & G_3 \\
  1 & 1 & 0 & 0 & 0 \\
  2 & 1 & 1 & 1 & 1 \\
  3 & 0 & 1 & 1 & 1 \\
  4 & 0 & 1 & 1 & 1 \\
\end{array} \]

A Classical endowment pattern

Here is an example of an endowment profile where this story breaks down

Terminology ‘Classical’ is from David Gale
Overlapping Generations

We are going to formalize this story and we will ask: When are Competitive Equilibria Pareto Efficient?
Definitions

Define

\[ C_s^t \quad \text{Consumption of generation } t \text{ at date } s \]

\[ \omega_s^t \quad \text{Endowment of generation } t \text{ at date } s \]

\[ p_t \quad \text{Money price of cookies at date } t \]
Eve solves

$$\max_{C_1^0} u^0(C_1^0), \quad \text{s.t.} \quad p_1 C_1^0 \leq 0$$

The solution to this problem is $C_1^0 = 0$
Competitive Equilibrium

Evelet and all subsequent generations solve

$$\max_{\{C_t^t, C_{t+1}^t\}} u(C_t^t, C_{t+1}^t) = \log(C_t^t) + \log(C_{t+1}^t)$$

$$p_t (1 - C_t^t) = a_{t+1}^t$$

$$p_{t+1} C_{t+1}^t = a_{t+1}^t$$

Where $a_{t+1}^t$ is the money value of savings/borrowing
Competitive Equilibrium

Consolidating the budget constraint

\[
\max_{\{c_t^t, c_{t+1}^t\}} \log(c_t^t) + \log(c_{t+1}^t)
\]

\[
c_t^t + c_{t+1}^t \frac{p_{t+1}}{p_t} \leq 1
\]
Competitive Equilibrium

The solution for all $\frac{p_{t+1}}{p_t} < \infty$ is,

$$C_t = \frac{1}{2}, \quad a_{t+1} = \frac{p_t}{2}, \quad C_{t+1} = \frac{1}{2 \frac{p_t}{p_{t+1}}}$$

Generation $t$ would like to lend to someone. But who do they lend to?

The only competitive equilibrium is

$$\frac{p_{t+1}}{p_t} = \infty, \quad C_t = 1$$
Competitive equilibrium is not Pareto Efficient

This is the competitive equilibrium allocation

\[ C_1^0 = 0 \]
\[ C_1^1 = 1 \]
\[ C_2^1 = 0 \]
\[ C_2^2 = 1 \]  
...  

Samuelson pointed out that Money can support this!

This is a Pareto dominating allocation

\[ C_1^0 = \frac{1}{2} \]
\[ C_1^1 = \frac{1}{2} \]
\[ C_2^1 = \frac{1}{2} \]
\[ C_2^2 = \frac{1}{2} \]  
...  

Why does the first welfare theorem fail?
A More general model from David Gale

\[ \omega_t^t = \omega^0 \]  
Endowment when young

\[ \omega_{t+1}^t = \omega^1 \]  
Endowment when old

\[ u(C_t^t, C_{t+1}^t) : \mathbb{R}_{++}^2 \rightarrow \mathbb{R} \]  
Utility function

\[ C_s^t \]  
Consumption of generation \( t \) in period \( s \)
Population Growth

David Gale allows for population growth

I will abstract from population growth to keep the algebra simpler
Consumer’s Problem, Generation 0

\[ \max_{C^0_1} u^0(C^0_1) \]

s.t.

\[ (C^0_1 - \omega^1) \leq \frac{M}{p_1} \]

Solution

\[ C^0_1 = \omega^1 + \frac{M}{p_1} \]

\( M \) is the fixed stock of fiat money.
Consumer’s Problem, Generation $t$

$$\max_{\{C_t^{t+1}, C_{t+1}^t\}} u^t(C_t^t, C_{t+1}^t)$$

s.t.

$$(\omega^0 - C_t^t) = \frac{M}{p_t}, \quad \quad (C_{t+1}^t - \omega^1) = \frac{M}{p_{t+1}}$$

The solution is characterized by a savings function $S$

$$\omega^0 - \hat{C}_t^t \equiv S\left(\frac{p_t}{p_{t+1}}\right), \quad \quad \omega^1 - \hat{C}_{t+1}^t = -\frac{p_t}{p_{t+1}} S\left(\frac{p_t}{p_{t+1}}\right)$$
Graphing the Solution

\[ S \left( \frac{p_t}{p_{t+1}} \right) \]

slope = \(-\frac{p_t}{p_{t+1}}\)
Equilibrium

Definition: A Competitive Equilibrium

i) a sequence of positive numbers \( \{p_t\}_{t=1}^{\infty} \)

ii) an allocation \( \{C_1^0\}, \{C_t^t, C_{t+1}^t\}_{t=1}^{\infty} \) such that

iv) \( S \left( \frac{p_1}{p_2} \right) = \frac{M}{p_1} \) from market clearing in period 1

v) \( S \left( \frac{p_t}{p_{t+1}} \right) = \frac{p_{t-1}}{p_t} S \left( \frac{p_{t-1}}{p_t} \right), \) from market clearing in period t
Equilibrium

If we define the inflation factor

\[ \pi_t = \frac{p_t}{p_{t-1}} \]

A competitive equilibrium is the solution to a difference equation

\[ S\left(\frac{1}{\pi_{t+1}}\right) = \frac{1}{\pi_t} S\left(\frac{1}{\pi_t}\right), \]
The Real Interest Rate

Let $R_t^N$ be the money interest factor (one plus the money interest rate) and define the real interest factor between dates $t$ and $t + 1$

$$R_{t+1} = \frac{R_t^N}{\pi_{t+1}}$$

Money does not pay interest and it is the only store of wealth. It follows that the real interest factor in this economy is the same as the inverse inflation factor

$$R_{t+1} = \frac{1}{\pi_{t+1}}$$
Equilibrium

Definition: A Stationary Competitive Equilibrium is

i) a competitive equilibrium where

\[
\frac{p_t}{p_{t+1}} = \frac{1}{\pi},
\]

\[(C^t_t, C^t_{t+1}) = (C^0, C^1) \quad \text{for all } t\]
Equilibrium

In a stationary competitive equilibrium

\[ S \left( \frac{1}{\pi} \right) = \frac{1}{\pi} S \left( \frac{1}{\pi} \right) \quad \Rightarrow \quad S \left( \frac{1}{\pi} \right) \left( 1 - \frac{1}{\pi} \right) = 0 \]

This implies that either

\[ \frac{1}{\pi} = 1, \quad \text{or} \quad S \left( \frac{1}{\pi} \right) = 0 \]
Theorem (Gale)

There are two types of stationary equilibria

Autarkic \[ S \left( \frac{1}{\pi} \right) = 0 \]

Golden Rule \[ \pi = 1 \]

When there is population growth, the golden rule implies that the real interest rate equals the population growth rate.

Samuelson called this the biological rate of interest
The Golden Rule (Phelps)

This is an example of what Edmund Phelps calls the golden rule of capital accumulation.

The golden rule states that a social planner could maximize steady state consumption by setting the real interest rate equal to the population growth rate.

If the interest rate is less than the growth rate the equilibrium is said to be dynamically inefficient.
Autarkic Equilibria

From 1st period market clearing

\[ S \left( \frac{1}{\pi} \right) = \frac{M}{p_1} = 0 \Rightarrow p_1 = \infty \]

Money has no value

But if agents are of different types there may still be borrowing and lending between members of the same generation
Golden Rule Equilibrium

From 1\textsuperscript{st} period market clearing

\[ S(1) = \frac{M}{p_1} \implies p_1 = \frac{M}{S(1)} \]

BUT: This equilibrium may not exist: Can you explain why?
Define: $\pi^{au}$ to be the solution to the equation

$$S\left(\frac{1}{\pi^{au}}\right) = 0.$$ 

Then $\frac{1}{\pi^{au}}$ is the **autarkic** steady state interest rate.

Define $\frac{1}{\pi^{gr}} = 1$ to be the **golden rule** steady state interest rate.
Definition

An economy is **Samuelson** if

$$\frac{1}{\pi au} < 1$$

An economy is **Classical** if

$$\frac{1}{\pi au} \geq 1$$
A Samuelson Economy

The two steady state equilibria in a Samuelson Economy

\[ S(1) > 0 \]

slope = \( 1 = \frac{1}{\pi gr} \)

slope = \( \frac{1}{\pi au} < 1 \)
There is only one feasible steady state equilibria in a Classical Economy.
Samuelson and Classical Economies

A Classical economy

No monetary equilibrium exists

\[ S\left(\frac{1}{\pi}\right) \]

\[ \frac{1}{\pi gr} = 1 \]

\[ \frac{1}{\pi} \]

A Samuelson economy

\[ S\left(\frac{1}{\pi}\right) \]

\[ \frac{1}{\pi au} \]

\[ \frac{1}{\pi gr} = 1 \]

\[ \frac{1}{\pi} \]
Why does the 1st Welfare Theorem Fail?

- Double infinity of agents and goods
- Pareto improving allocation is feasible but involves infinite wealth at equilibrium prices
- Shell “Notes on the Economics of Infinity” *JPE* 1971
An Example

\[ U = \log(C_t^t) + \beta \log(C_{t+1}^t), \quad C_t^t + \frac{C_{t+1}^t}{\pi_{t+1}} \leq \omega^0 + \frac{\omega^1}{\pi_{t+1}} \]

Solution

\[ S \left( \frac{1}{\pi_{t+1}} \right) = \frac{\beta \omega^0}{1 + \beta} - \frac{1}{1 + \beta \pi_{t+1}} \omega^1, \]

\[ C_t^t = \frac{1}{1 + \beta} \left( \omega^0 + \pi_{t+1} \omega^1 \right), \quad C_{t+1}^t = \frac{1}{1 + \beta} \left( \frac{\omega^0}{\pi_{t+1}} + \omega^1 \right) \]
These non-stationary sequences are all valid equilibria. This is the only equilibrium.

The Samuelson case has two valid stationary equilibria.

The classical case has only one.

\[
\frac{1}{\pi_{t+1}} = \frac{\omega^1}{\omega^1 + \omega^0 \beta \left(1 - \frac{1}{\pi_t}\right)},
\]

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## Summary of Equilibria in the OG Model

<table>
<thead>
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<th><strong>Samuelson</strong></th>
<th><strong>Classical</strong></th>
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<tbody>
<tr>
<td><strong>Stationary</strong></td>
<td><strong>Exists and is Pareto Inefficient</strong></td>
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<tr>
<td><strong>Autarkic</strong></td>
<td></td>
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<tr>
<td><strong>Stationary</strong></td>
<td><strong>Exists: Money has value: is Pareto Efficient</strong></td>
<td><strong>Doesn't exist (would need negative money)</strong></td>
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<tr>
<td><strong>Monetary</strong></td>
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<tr>
<td><strong>Non-Stationary</strong></td>
<td><strong>Exists: All converge to autarky. All are Pareto Inefficient</strong></td>
<td><strong>Don't exist</strong></td>
</tr>
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<td><strong>Monetary</strong></td>
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Equilibrium: a Definition

An allocation, $x$, is an $\ell \times m$ matrix that specifies how much of commodity $i \in \{1, \ldots, \ell\}$ goes person $j \in \{1, \ldots, m\}$

A competitive equilibrium $e$ is a price vector $\hat{p}$ and allocation $\hat{x}$ such that $\hat{x}$ is chosen optimally at prices $\hat{p}$.

$$e \equiv \{\hat{p}, \hat{x}\}$$
1) An equilibrium \( e = (\hat{p}, \hat{x}) \) is indeterminate if there exists an \( \varepsilon > 0 \) such that for all \( \delta < \varepsilon \) there is another equilibrium \( e' = (\hat{p}'(\delta), \hat{x}'(\delta)) \) such that \( ||e' - e|| < \delta \)

2) An equilibrium is determinate if it is not indeterminate.
Determinacy and Indeterminacy

Indeterminacy is non-generic (almost never happens) in finite GE models

Indeterminacy is generic (happens often) in infinite horizon economies whenever the welfare theorems fail

The overlapping generations model is an example of this
Dierker’s Theorem in 3D

• In a finite Exchange economy there is (generically) a finite odd number of equilibria

The theorem says that this kind of behavior is non-generic

\[ f^1, f^2 \]

A set of indeterminate equilibria

\[ S^3(p) \]
Dynamic determinacy

This is an example of a determinate steady state sequence $\{\hat{x}_t, \hat{p}_t\}_{t=1}^\infty$

This is the equilibrium sequence that begins at $\{x_0, p_0\} = \{\bar{x}_0, \bar{p}_0\}$
Here, $\hat{p}_t$ is an equilibrium but $\{p^1_t\}$ and $\{p^2_t\}$ are not. They violate boundedness.
When the system has too many stable eigenvalues, there are multiple dynamic paths consistent with boundedness.
Here, $\{\hat{p}_t\}$ is an equilibrium and so are $\{p^1_t\}$ and $\{p^2_t\}$. They are all bounded.
Summary

• The equilibria of overlapping generations models are the solutions to difference equations
• Equilibria may be dynamically inefficient
• Equilibria are of two types: Samuelson and Classical