Partial Equilibrium Search Theory

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Road map

1. The prototypical, stationary search model
2. On-the-job search and layoffs
3. Non-stationary search
1. The prototypical, stationary search model
Assumptions

- Time is continuous and individuals live for ever.
- In a small time interval $[t, t + \Delta]$, unemployed individuals receive a payment $b\Delta$.
  - Parameter $b$ is the opportunity flow-cost of employment (including unemployment insurance, search costs, etc.).
- When unemployed, the probability of receiving an offer in $[t, t + \Delta]$ is $\lambda_0\Delta$, irrespective of the time already spent waiting for an offer. Such a point process is called a Poisson process.
- Employment lasts for ever.
- The distribution of wage offers is $F$ and $F$ is continuous and differentiable.
- The time-discount rate is $\rho > 0$. 
Poisson process

- Spell durations are independent: if $0, t_1$ and $t_2$ are three point realizations (three subsequent job offers), then $t_1$ and $t_2$ are independent.

- A spell duration has an exponential distribution with parameter $\lambda_0$:
  - the density is: $f_0(t) = \lambda_0 \exp(-\lambda_0 t)$
  - the cdf is: $F_0(t) = 1 - \exp(-\lambda_0 t))$
  - the expectation is: $\int_0^\infty t \lambda_0 e^{-\lambda_0 t} dt = \frac{1}{\lambda_0}$

- **Important property:** At any time $t$, the elapsed time duration separating $t$ from the last point realization, say $\tau_0$, and the residual time duration separating $t$ from the next point realization, say $\tau_1$, are independent and have the same marginal distribution: an exponential distribution with parameter $\lambda_0$. 
Employment value function

- Let \( W(w) \) be the value of a job offering a wage \( w \) per unit of time \( \Delta \).
- **First calculation.** Assuming that employment spells last for ever, 
  \[
  W(w) = \int_0^\infty we^{-\rho t} dt = \frac{w}{\rho}
  \]
- **Second calculation.** Remark that value functions do not depend on calendar time because of the stationary environment: the value tomorrow is equal to the value today.

  Bellman’s optimality principle yields 
  \[
  W(w) = \frac{w\Delta}{1 + \rho\Delta} + \frac{1}{1 + \rho\Delta} W(w)
  \]

  \( \Leftrightarrow (1 + \rho\Delta) W(w) = w\Delta + W(w) \)

  \( \Leftrightarrow W(w) = \frac{w}{\rho} \)
Let $V_0$ be the value of unemployment.
Unemployed workers accept any wage offer $w$ such that $W(w) \geq V_0$.
Bellman’s principle yields

$$V_0 = \frac{b\Delta}{1 + \rho\Delta} + \frac{\lambda_0\Delta}{1 + \rho\Delta} \mathbb{E}_{w \sim F} \max \{W(w), V_0\} + \frac{1 - \lambda_0\Delta}{1 + \rho\Delta} V_0$$

$$\Leftrightarrow \rho V_0 = b + \lambda_0 \mathbb{E}_{w \sim F} \max \{W(w) - V_0, 0\}$$

Option-value equation:

$$\rho V_0 = b + \lambda_0 \mathbb{E}_{w \sim F} \max \{W(w), V_0\} - V_0$$

and

$$\rho W = w$$

no option
Unemployment value function 2

- $W(w) > V_0 \iff w > \rho V_0$ (reservation wage)
- Hence,

$$E_{w \sim F} \max \{ W(w) - V_0, 0 \} = E_{w \sim F} \max \left\{ \frac{w}{\rho} - V_0, 0 \right\}$$

$$= \int_{\rho V_0}^{\bar{w}} \left( \frac{w}{\rho} - V_0 \right) dF(w)$$

(integration by part) $$= - \left[ \left( \frac{w}{\rho} - V_0 \right) \bar{F}(w) \right]_{\rho V_0}^{\bar{w}} + \frac{1}{\rho} \int_{\rho V_0}^{\bar{w}} \bar{F}(w) dw$$

$$= \frac{1}{\rho} \int_{\rho V_0}^{\bar{w}} \bar{F}(w) dw$$

where $\bar{F}(w) \equiv 1 - F(w)$.
Reservation wage policy

- The optimal strategy when unemployed is to accept any job offering a wage \( w \) such that
  \[ W(w) \geq V_0 \iff w \geq \phi \equiv \rho V_0 \]

- The reservation wage \( \phi \) is the solution to the equation:
  \[ \phi = b + \frac{\lambda_0}{\rho} \int_{\phi}^{\bar{w}} \bar{F}(w) \, dw \]

- Remarks on computation:
  - The integral usually has no closed form. A numerical procedure is required to approximate the integral (quadrature, Simpson, etc.), available in softwares like R, Gauss or Matlab.
  - The equation can then be solved for \( \phi \), also numerically, using a Newton-type algorithm.
Labor force surveys usually observe workers continuously during a time interval \([t_0, t_1]\) and gather retrospective information at \(t_0\) so that, for unemployed workers at \(t_0\), it is possible to record:

- the elapsed unemployment duration at \(t_0\): \(\tau_0\)
- the residual unemployment duration after \(t_0\): \(\tau_1\)
- the accepted wage \(w\) at \(t_0 + \tau_1\) if the worker leaves unemployment by the end of the recording period \(t_1 - t_0\)
The duration of unemployment has an exponential distribution with parameter $\lambda_0 \bar{F}(\phi)$ (instantaneous probability of receiving an offer times the probability that it is acceptable).

The Poisson property implies that $\tau_0$ and $\tau_1$ are independent and exponentially distributed.

The density of $(\tau_0, \tau_1, w)$ is therefore equal to

$$
\ell(\tau_0, \tau_1, w) = \lambda_0 \bar{F}(\phi) \exp (-\lambda_0 \bar{F}(\phi) \tau_0) \times \left[ \lambda_0 \bar{F}(\phi) \exp (-\lambda_0 \bar{F}(\phi) \tau_1) \cdot \frac{f(w)}{\bar{F}(\phi)} \right]^z [\exp (-\lambda_0 \bar{F}(\phi) \tau_1)]^{1-z}
$$

Here $z = 1$ if $\tau_1 < t_1 - t_0$, $z = 0$ if $\tau_1 = t_1 - t_0$ (the residual unemployment duration is censored).

The distribution of wage offers is identified only conditional on $w > \phi$ (with density $\frac{f(w)}{\bar{F}(\phi)}$).
Estimation - First approach

- Treat reservation wage $\phi$ as a parameter.
- Maximize the likelihood of a sample $\{(\tau_{0i}, \tau_{1i}, w_i, z_i), i = 1, \ldots, N\}$ subject to $w_i \geq \phi, \forall i$.
- The reservation wage $\phi$ is estimated by the minimal accepted job offer: $\hat{\phi} = \min \{w_i \mid z_i = 1\}$.
  - It is a superconsistent estimator: $N(\hat{\phi} - \phi) \overset{L}{\rightarrow}$ non Gaussian distribution.
- Estimate $\lambda_0$ and the parameters of $F$ by maximising the log likelihood conditional on $\phi = \hat{\phi}$ ($\hat{\phi}$ is like a constant as it is superconsistent);
- estimate $b$ as $\hat{b} = \hat{\phi} - \frac{\hat{\lambda}_0}{\rho} \int_{\hat{\phi}}^{\infty} \overline{F}(w)dw$.
- One can cluster the sample by individual type.
- For a complete treatment see Christensen and Kiefer (1991)
Estimation - Second approach

- Same but do ML subject to reservation wage equation (gets you sd for $b$).
- Add conditioning variables (determining $b, F$ or $\lambda_0$) if necessary.
2. On-the-job search and layoffs

- Mortensen and Neumann (1989)
- I also deal with wage offer distribution $F$ that may exhibit mass points or be discrete (important for the sequel).
Cumulative Distribution Functions

Let $F : \mathbb{R} \to [0, 1]$ be the CDF of a random variable $X$.

- It is cadlag. “Continue à droite, limite à gauche.” Right continuous with left limits.
- It has at most countably many discontinuities $\{x_1, x_2, \ldots\}$ and $\Pr\{X = x_i\} = F(x_i) - F(x_i-) \equiv \Delta F(x_i)$.
- It has a left-derivative $F'(x) = \lim_{\varepsilon \downarrow 0} \frac{F(x) - F(x-\varepsilon)}{\varepsilon}$ at almost every point.
- $F_c(t) = \int_{-\infty}^{t} F'(x) \, dx$ is the absolutely continuous part of $F$ and $F_s(t) = F(t) - F_c(t) = \sum_{x \in (0,t]} \Delta F(x)$ is the singular part:

$$F(t) = F(0) + \int_{0}^{t} F'(x) \, dx + \sum_{x \in (0,t]} \Delta F(x)$$
Lebesgue-Stieltjes measure and integral

- **Lebesgue-Stieltjes measure.** This is the Borel measure $\mu$ on $\mathbb{R}$ such that
  
  1. $\mu(a, b] = F(b) - F(a)$
  2. $\mu[a, b] = F(b) - F(a-)$
  3. $\mu[a, b) = F(b-) - F(a-)$
  4. $\mu(a, b) = F(b-) - F(a)$
  5. $\mu\{a\} = F(a) - F(a-) = \Delta F(a)$.

- **Stieltjes integral.** For any Borel-measurable real-valued function $g$ and any Borel set $B$ (union of intervals), define $\int_B g \, dF$ to be $\int_B g \, d\mu$, with the convention that $\int_{a}^{b} g \, dF = \int_{(a,b]} g \, dF$. 

Stieltjes integral

- For non pathological CDFs on \([0, +\infty)\),
  \[
  \int_a^b g \, dF = \int_a^b g(x)F'(x) \, dx + \sum_{x \in (a,b]} g(x)\Delta F(x)
  \]

- **Integration by part.** Given two non-decreasing functions (or of bounded variation on any finite integral), which are both cadlag, then
  \[
  F(b)G(b) - F(a)G(a) = \int_a^b F(x-)dG(x) + \int_a^b G(x)dF(x)
  = \int_a^b F(x-)dG(x) + \int_a^b G(x-)dF(x) + \sum_{x \in (a,b]} \Delta F(x)\Delta G(x)
  \]

  - See e.g. Fleming, Harrington (2005), *Counting Processes and Survival Analysis*, John Wiley & Sons, Inc.
Assumptions

- When employed, job offers accrue at constant rate $\lambda_1$. (It is likely that $\lambda_1 < \lambda_0$.)
- Jobs are exogenously destroyed at constant rate $\delta$.
- Employees at wage $w$ accept an alternative job paid $x$ if only $W(x) > W(w)$. 
The unemployment value function proceeds from the same definition as before,

\[ \rho V_0 = b + \lambda_0 \mathbb{E}_{w \sim F} \max \{ W(w) - V_0, 0 \} \]

\[ = b + \int_{\phi}^{\bar{w}} [W(w) - V_0] dF(w) \]
Employment value

The employment value is

\[ W(w) = \frac{w \Delta}{1 + \rho \Delta} + \frac{\lambda_1 \Delta}{1 + \rho \Delta} \mathbb{E}_{x \sim F} \max \{ W(x), W(w) \} \]

\[ + \frac{\delta \Delta}{1 + \rho \Delta} V_0 + \frac{1 - \delta \Delta - \lambda_1 \Delta}{1 + \rho \Delta} W(w) \]

that is

\[ \rho W(w) = w + \lambda_1 \mathbb{E}_{x \sim F} \max \{ W(x) - W(w), 0 \} + \delta [V_0 - W(w)] \]

\[ = w + \lambda_1 \int_{w}^{\bar{w}} [W(x) - W(w)] dF(x) + \delta [V_0 - W(w)] \]
$W$ is continuous for all $F$

- For all real $z, x$ and $y$,
  \[
  |\max\{z, x\} - \max\{z, y\}| \leq |x - y|.
  \]

- Moreover, for all couple of rv $X, Y$, $|\mathbb{E}X - \mathbb{E}Y| \leq \mathbb{E}|X - Y|$. Hence
  \[
  |\mathbb{E}_W \max\{W(w), W(x)\} - \mathbb{E}_W \max\{W(w), W(y)\}| \\
  \leq \mathbb{E}_W |\max\{W(w), W(x)\} - \max\{W(w), W(y)\}| \\
  \leq \mathbb{E}_W |W(x) - W(y)| = |W(x) - W(y)|
  \]

- Bellman equation
  \[
  (\rho + \lambda_1 + \delta) W(w) = w + \lambda_1 \mathbb{E}_{x \sim F} \max\{W(x), W(w)\} + \delta V_0
  \]
  implies that
  \[
  (\rho + \delta) |W(x) - W(y)| \leq |x - y|
  \]
  and so $W$ is Lipschitz-continuous.
$W$ is increasing for all $F$

- For all $k$ and $x < y$,
  
  \[
  \max\{k, W(x)\} - \max\{k, W(y)\} \geq \min\{W(x) - W(y), 0\}.
  \]

- Then,
  
  \[
  (\rho + \lambda_1 + \delta)[W(x) - W(y)] \geq x - y + \lambda_1 \min\{W(x) - W(y), 0\}.
  \]

- If $W(x) - W(y) \leq 0$ then
  
  \[
  (\rho + \delta)[W(x) - W(y)] \geq x - y > 0,
  \]

  yielding a contradiction. Hence $W(x) > W(y)$.
Integration by part

Because $W$ is continuous,

- **Unemployment value:**

$$\int_{\phi}^{W} [W(w) - V_0] \, dF(w) = \int_{\phi}^{W} \overline{F}(w) \, dW(w)$$

- **Employment value:**

$$\int_{w}^{W} [W(x) - W(w)] \, dF(x) = \int_{w}^{W} \overline{F}(x) \, dW(x)$$
$W$ is left-differentiable

- This function is differentiable (because $W$ is continuous – like a continuous cdf) and its left-derivative (like a pdf) is equal to $-\bar{F}(w^-)W'(w)$, where $W'(w)$ is the left-derivative of $W(w)$.

- Indeed

$$\int_{w}^{w} - \int_{w-\varepsilon}^{w} = \int_{(w,w]} - \int_{(w-\varepsilon,w]} = \int_{(w-\varepsilon,w]}$$

- On the interval $(w-\varepsilon, w)$ and small $\varepsilon$ the value of $F(x)$ is close to $\bar{F}(w^-) = \Pr\{\text{wage} > w\}$ assuming that the jump points of $F$ are not dense.
\( W \) is left-differentiable

- Using the Bellman equation for \( W(x) \), the left-derivative of \( W \) is obtained as

\[
W'(x) = \frac{1}{\rho + \delta + \lambda_1 \overline{F}(x^-)} > 0
\]

(Left- and right-derivatives differ at points of discontinuity of \( F \).)

- Then,

\[
\int_{w}^{\overline{w}} \overline{F}(x) dW(x) = \int_{w}^{\overline{w}} \overline{F}(x) W'(x) dx = \int_{w}^{\overline{w}} \frac{\overline{F}(x)}{\rho + \delta + \lambda_1 \overline{F}(x)} dx
\]

(The discontinuities don’t matter inside the Riemann integral because they are at most countably many.)
Reservation wage

- Unemployment value:

\[ \rho V_0 = b + \lambda_0 \int_\phi^w F(w) dW(w) = b + \lambda_0 \int_\phi^w \frac{F(w)}{\rho + \delta + \lambda_1 F(w)} dw \]

- Employment value:

\[ (\rho + \delta) W(w) = w + \delta V_0 + \lambda_1 \int_w^w \frac{F(x)}{\rho + \delta + \lambda_1 F(x)} dx \]

- Reservation wage:

\[ W(\phi) = V_0 \iff \phi = b + (\lambda_0 - \lambda_1) \int_\phi^w \frac{F(w)dw}{\rho + \delta + \lambda_1 F(w)} \]

- Note that \( \lambda_0 > \lambda_1 \Rightarrow \phi > b \). Because of low wages, \( b \) is often estimated negative.
Non-stationary search

- Van den Berg (1990)
Assumptions

- $b$ now depends on unemployment duration (or exit rates, ...)
- For simplicity, we rule out on-the-job search and layoffs, so that the employment value continues to be $\mathcal{W}(w) = \frac{w}{\rho}$. 
Unemployment value

- Let $V_0(t)$ denote the value of unemployment for unemployment duration $t$.
- Bellman equation:

$$V_0(t) = \frac{b(t)\Delta}{1 + \rho\Delta} + \frac{\lambda_0\Delta}{1 + \rho\Delta} \mathbb{E} \max \{ W(w), V_0(t + \Delta) \} + \frac{1 - \lambda_0\Delta}{1 + \rho\Delta} V_0(t + \Delta)$$

Hence,

$$\rho V_0(t) = b(t) + \lambda_0 \mathbb{E}_{w \sim F} \max \{ W(w) - V_0(t + \Delta), 0 \} + \frac{V_0(t + \Delta) - V_0(t)}{\Delta}$$

$$= b(t) + \lambda_0 \int_{\rho V_0(t)}^{\overline{w}} \left( \frac{w}{\rho} - V_0(t) \right) dF(w) + \frac{V_0(t + \Delta) - V_0(t)}{\Delta}$$

$$= b(t) + \frac{\lambda_0}{\rho} \int_{\rho V_0(t)}^{\overline{w}} \overline{F}(w) dw + \frac{V_0(t + \Delta) - V_0(t)}{\Delta}$$

- This is an integral-differential equation that can be solved backward assuming stationarity for $t > T$ ($V_0(t) = V_0(T), \forall t > T$).
Reservation wage

- $\phi(t) = \rho V_0(t)$
- If $b(t)$ is decreasing then so is the reservation wage. Workers are less picky as they approach the end of the insurance period.
References

