Wednesday Lecture 1
Leverage Cycles

August 8, 2012
Overview

- A general equilibrium model of the role of leverage and “haircuts” in asset markets
- Endogenous loan contracts determine “haircuts” and interest rates
- “Haircuts” are an important market-clearing variable
- Explains why small changes in beliefs about the fundamental values of assets can cause violent drops in asset prices
- Importance of heterogeneous beliefs and “natural owners”
Assumptions:

- There are two dates, $t = 0, 1$, and two terminal states, $U, D$ (agents have heterogeneous beliefs)
- There is a single good that can be consumed immediately or stored costlessly
- There is a single asset. One unit of the asset has a payoff of $u$ (resp. $d$) units of the good in state $U$ (resp. $D$)
- There is a continuum of risk neutral, patient (no discounting) agents, where
  - each agent has one unit of the good at date 0 (and nothing at date 1) and one unit of the asset;
  - agents are parameterized by their probability belief $\alpha \in [0, 1]$ that state $U$ will occur;
  - beliefs are uniformly distributed: $\alpha \sim U [0, 1]$
- Short sales are not allowed
Equilibrium without borrowing

- Ownership of the asset will be more dispersed if borrowing is not allowed
- Agent $\alpha$ will demand the asset if
  \[ \alpha u + (1 - \alpha) d > p \]

  The marginal buyer $\alpha^*$ is defined by the condition
  \[ \alpha^* u + (1 - \alpha^*) d = p \] (1)

- The asset market clears at $p^*$ if
  \[ 1 - \alpha^* = p^* \alpha^* \] (2)

  Substituting (1) into (2) gives us
  \[ 1 - \alpha^* = (\alpha^* u + (1 - \alpha^*) d) \alpha^* \]
Numerical example

- Let \( u = 1 \) and \( d = 0.2 \)
- Then the marginal buyer \( \alpha^* \) satisfies

\[
1 - \alpha^* = (\alpha^* + (1 - \alpha^*) 0.2) \alpha^*
\]

\[
\Rightarrow \alpha^* = 0.59629
\]

- Substituting this value into (1) gives us

\[
p^* = \alpha^* + (1 - \alpha^*) (0.2)
\]

\[
= 0.67703,
\]

- In other words, it takes over 40% of the population to hold the assets, which they purchase at a price of \( p^* = 0.677 \)
Numerical example

- Without loss of generality we can assume that all consumption occurs at date 1; then agents with $\alpha > \alpha^*$ have equilibrium asset holdings

\[
(x_\alpha, y_\alpha) = \left(0, 1 + \frac{1}{p^*}\right) \approx (0, 2.477)
\]

at the end of date 0 and consume

\[
c_\alpha = (c_\alpha U, c_\alpha D) \approx (2.477, 0.4954)
\]

- Similarly, agents with $\alpha < \alpha^*$ have equilibrium asset holdings

\[
(x_\alpha, y_\alpha) = (1 + p^*, 0) \approx (1.677, 0)
\]

at the end of date 1 and consume

\[
c_\alpha = (c_\alpha U, c_\alpha D) \approx (1.677, 1.677)
\]
Now suppose agents can borrow if they provide collateral for the debt: the collateral constraint takes the form

\[ b_\alpha \leq (0.2) y_\alpha, \]

where \( b_\alpha \) is the face value of the debt and \( y_\alpha \) is the total amount of the asset held by agent \( \alpha \).

Note that all asset holdings are used as collateral, not just the assets funded by debt.

The price of debt is denoted by \( q_0 \), so the amount borrowed is \( q_0 b_\alpha \).

The debt is risk-free and pays \( b_\alpha \) in each state; in equilibrium, debt must be priced at par, \( q_0 = 1 \).
Equilibrium borrowing

- Let $\alpha^*$ be the marginal buyer defined by
  \[
  \alpha^* + (1 - \alpha^*) (0.2) = p^* \tag{3}
  \]
- This agent is indifferent between holding cash and using it to buy assets
- One unit of the asset bought on margin costs $p^* - 0.2$ and the return is $\alpha^* (1 - 0.2)$. Hence, the return per dollar is
  \[
  \frac{\alpha^* (1 - 0.2)}{p^* - 0.2} = \frac{\alpha^* (1 - 0.2)}{\alpha^* + (1 - \alpha^*) (0.2) - 0.2} = 1
  \]
- Thus, agent $\alpha^*$ is also indifferent between holding cash and buying the asset on margin
Agent \( \alpha > \alpha^* \) strictly prefers to hold the asset, purchased for cash or bought on margin, but he makes a higher return buying on margin than paying cash:

\[
\frac{\alpha (1 - 0.2)}{p^* - 0.2} = \frac{\alpha (1 - 0.2)}{\alpha^* + (1 - \alpha^*) (0.2) - 0.2}
\]

\[
= \frac{\alpha}{\alpha^*} > \frac{\alpha + (1 - \alpha) (0.2)}{\alpha^* + (1 - \alpha^*) (0.2)}
\]

\[
= \frac{\alpha + (1 - \alpha) (0.2)}{p^*}
\]

Similarly, an agent \( \alpha < \alpha^* \) makes a lower return buying on margin than paying cash.
Market clearing

- Market clearing requires
  \[
  \frac{1 - \alpha^* + 0.2}{\alpha^*} = p^*
  \]  
  \[\text{(4)}\]

- Note that the total amount of money borrowed is 0.2 because the buyers \(\alpha \in [\alpha^*, 1]\) have one unit of collateral.

- Combining (3) and (4) we get
  \[
  \frac{1 - \alpha^* + 0.2}{\alpha^*} = p^* = \alpha^* + (1 - \alpha^*) (0.2)
  \]

- The solution is \(\alpha^* = 0.68614\), so that
  \[
  p^* = \alpha^* + (1 - \alpha^*) (0.2) = 0.74891
  \]

- The asset can be purchased by just 31% of the population (the natural buyers) even though the price has risen to 0.75 (compared to 40% and 0.68, respectively).
Equilibrium leverage

- Why is the collateral constraint

\[ b_\alpha \leq (0.2) y_\alpha? \]

- Consider a debt contract \((b, 1)\), where \(b\) is the face value of the promise and 1 is the amount of collateral (w.l.o.g., normalize collateral to one, since all contracts are homogeneous of degree one)

- The debt contract \((b, 1)\) will pay \(\min \{ b, 1 \}\) in state \(U\) and \(\min \{ b, 0.2 \}\) in state \(D\)

- We can describe an equilibrium with a range of contracts \(\{ b, 1 \}\) and price the debt corresponding to each level

- In equilibrium, only debt with an 80% haircut will be traded
Equilibrium leverage

- Contracts like \((b, 1)\) with \(b < 0.2\) are over-collateralized and should never be used.

- Contracts with \(b > 1\) are equivalent to contracts with \(b \leq 1\); thus, consider contracts with \(0.2 \leq b \leq 1\).

- For any \(0.2 < b \leq 1\), the contract \((b, 1)\) promises \(b > 0.2\) in state \(U\) and 0.2 in state \(D\). Then

\[
\alpha b + (1 - \alpha) (0.2) \leq \alpha^* b + (1 - \alpha^*) (0.2) \quad \text{as} \quad \alpha \leq \alpha^*,
\]

i.e., the contract is more highly valued by \(\alpha > \alpha^*\) relative to the contract \((0.2, 1)\) and less highly valued by \(\alpha < \alpha^*\).

- Thus, agents \(\alpha > \alpha^*\) will only supply this contract at a price \(q_j > q_0 = 1\) and agents \(\alpha < \alpha^*\) will only demand it at a price \(q_j < q_0 = 1\); so trade in these contracts cannot occur in equilibrium.
Applications

Violations of the Law of One Price

- Suppose there are two assets, Red and Blue, with the same payoffs, but Red assets can be used as collateral and Blue cannot (why not?)
- The prices of Red and Blue assets will not be equal (Red asset prices will be higher)
- In equilibrium, the Red assets will be held by the most optimistic agents without collateral, the Blue assets will be held by less optimistic individuals, and the least optimistic individuals will sell their assets

Legacy vs new assets

- When markets freeze, should government provide accept legacy assets as collateral or only new assets?
- Geanakoplos shows that accepting all assets will raise the prices of new assets as well
A three-period model

- The previous example is extended to allow for three dates $t = 0, 1, 2$ and four states $UU, UD, DU$ and $DD$ with asset payoffs equal to 1 after $UU, UD, DU$ and 0.2 after $DD$
There is a continuum of risk neutral and patient agents \( \alpha \in [0, 1] \), each with an endowment of one unit of the good and one unit of the asset at date 0 and nothing at subsequent dates.

One period loans (repos) with non-recourse collateral are used to finance purchases of the assets. As before, the only loans traded are those that promise the maximal face value that is risk free, i.e., that promise

$$b_s = \min \{p_{sU} + d_{sU}, p_{sD} + d_{sD}\}$$

in states \( s = 0, U, D \).

The risk neutral interest rate is assumed to be 0

W.l.o.g., we can assume that all consumption takes place at the last date.
Equilibrium in state D at date 1

Let \( \alpha^* \) be the marginal buyer at date 0 and let \( \beta^* \) be the marginal buyer in state \( D \) at date 1

Then

\[
p_D = \beta^* + (1 - \beta^*) (0.2) \tag{5}
\]

implies that \( \beta^* \) is indifferent between buying the asset and holding goods. This implies that he is indifferent between buying on margin and holding goods (see section on collateral)

The price \( p_D \) is determined by the equation

\[
p_D = \frac{1}{\alpha^*} (\alpha^* - \beta^*) + 0.2 \tag{6}
\]

The agents in \([\alpha^*, 1]\) are all out of business and have sold their assets to pay their debts; the agents \( \alpha \in [0, \alpha^*] \) hold all the goods and assets in equal amounts; thus each of them has \( 1/\alpha^* \) units of goods and assets
Equilibrium at date 0

- At date 0, the price of the asset is equal to the ratio of goods supplied by agents \( \alpha \in [\alpha^*, 1] \) to the number of assets supplied by agents \( \alpha \in [0, \alpha^*] \):

\[
p_0 = \frac{1 - \alpha^* + p_D}{\alpha^*}
\] (7)

- The marginal agent at date 0 levers \( p_0 - p_D \) dollars to obtain \( 1 - p_D \) dollars in state \( U \) and nothing in state \( D \), so the return per dollar is

\[
\frac{\alpha^* (1 - p_D)}{p_0 - p_D}
\]

- One dollar stored until date 1 is worth 1 in state \( U \), but in state \( D \) he could buy the asset on margin and earn

\[
\frac{\alpha^* (1 - 0.2)}{p_D - 0.2} = \frac{\alpha^* (1 - 0.2)}{\beta^* + (1 - \beta^*) (0.2) - 0.2} = \frac{\alpha^*}{\beta^*}
\]

Thus, the return to storing one dollar is

\[
\alpha^* + (1 - \alpha^*) \frac{\alpha^*}{\beta^*}
\]
Equilibrium at date 0

- Equating the returns to holding a dollar and buying the asset on margin at date 0,
  \[
  \frac{\alpha^* (1 - p_D)}{p_0 - p_D} = \alpha^* + (1 - \alpha^*) \frac{\alpha^*}{\beta^*}
  \]  
  \[\text{(8)}\]

- Finally, if the marginal buyer purchases the asset with cash, he pays \(p_0\) and receives 1 with probability \(\alpha^*\) and \(p_D\) with probability \(1 - \alpha^*\)

- A payment of \(p_D\) in state \(D\) is worth \(p_D \frac{\alpha^*}{\beta^*}\) to \(\alpha^*\)

- From the equation (8) we get
  \[
  \frac{\alpha^* (1 - p_D)}{p_0 - p_D} = \frac{p_D \left( \alpha^* + (1 - \alpha^*) \frac{\alpha^*}{\beta^*} \right)}{p_0 - p_D + p_D} \\
  = \frac{\alpha^* (1 - p_D) + p_D \left( \alpha^* + (1 - \alpha^*) \frac{\alpha^*}{\beta^*} \right)}{p_0 - p_D + p_D} \\
  = \frac{\alpha^* + (1 - \alpha^*) \frac{\alpha^*}{\beta^*}}{p_0}
  \]
Solution

- Thus we have shown that $\alpha^*$ and $\beta^*$ are the marginal buyers at date 0 and at date 1 in state $D$.
- Equilibrium is determined by the equations (5), (6), (7) and (8).
- The solution to this system of equations is:
  
  \[
  p_0 = 0.95, \\
  \alpha^* = 0.86667, \\
  p_D = 0.69, \\
  \beta^* = 0.35
  \]

- Thus, in the three-period model, the price is higher in the first period and is held by a smaller number of agents compared to the two-period model, whereas in the second period it falls to a lower price than in the isomorphic two-period model.