Monday Lecture 2
Optimal Intermediation

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Bryant (1980) and Diamond and Dybvig (1983) made four contributions:

(i) A maturity structure of bank assets, in which less liquid assets earn higher returns
(ii) A theory of liquidity preference, modeled as uncertainty about the timing of consumption
(iii) The representation of a bank as an intermediary that provides insurance to depositors against liquidity (preference) shocks
(iv) An explanation of bank runs by depositors

In the case of Diamond and Dybvig (1983), the bank runs are modeled as the result of self-fulfilling prophecies or panics; in the case of Bryant (1980), they are modeled as the result of fundamentals
The basic setup

- There are three dates, \( t = 0, 1, 2 \). At each date there is an all-purpose consumption/investment good.
- There are two types of assets:
  - the liquid asset (also called the short asset) is a constant returns to scale technology that takes one unit of the good at date \( t \) and converts it into one unit of the good at date \( t + 1 \), where \( t = 0, 1 \);
  - the illiquid asset (also called the long asset) is a constant returns to scale technology that takes one unit of the good at date 0 and transforms it into \( R > 1 \) units of the good at date 2; if the long asset is liquidated prematurely at date 1 then it pays \( 0 < r < 1 \) units of the good for each unit invested.
- There is a large number of ex ante identical economic agents. Each consumer has an endowment of one unit of the good at date 0 and nothing at the later dates.
Consumers

- With probability $\lambda$ an agent is an **early consumer**, who only values consumption at date 1; with probability $(1 - \lambda)$ he is a **late consumer** who only values consumption at date 2. The agent’s (random) utility function $u(c_1, c_2)$ is defined by

$$u(c_1, c_2) = \begin{cases} U(c_1) & \text{w.pr. } \lambda \\ U(c_2) & \text{w.pr. } 1 - \lambda, \end{cases}$$

where $c_t \geq 0$ denotes consumption at date $t = 1, 2$ and $U(\cdot)$ is a neoclassical utility function (increasing, strictly concave, twice continuously differentiable)

- “Law of large numbers”: $\lambda$ is the fraction of early consumers; $1 - \lambda$ is the fraction of late consumers

- Uncertainty about the agent’s type gives rise to **liquidity preference**
Market equilibrium

- We assume that there exists a market on which an agent can sell his holding of the long asset at date 1 after he discovers his true type.
- At date 0, a consumer invests in a portfolio \((x, y)\) subject to the budget constraint
  \[
  x + y \leq 1
  \]
- At date 1 he discovers his type. An early consumer liquidates his portfolio and consume the proceeds:
  \[
  c_1 = y + Px
  \]
  where \(P\) is the price of the long asset.
- A late consumer rebalance his portfolio (w.l.o.g., we assume he holds only the long asset):
  \[
  c_2 = \left( x + \frac{y}{P} \right) R
  \]
- At date 0, the consumer chooses \((x, y)\) to maximize expected utility
  \[
  \lambda U (y + Px) + (1 - \lambda) U \left[ \left( x + \frac{y}{P} \right) R \right]
  \]
In equilibrium, the price of the long asset must be $P = 1$, so the consumer’s consumption is

$$c_1 = x + Py = x + y = 1$$

at date 1 and

$$c_2 = \left(x + \frac{y}{P}\right)R = (x + y)R = R$$

at date 2

Then the equilibrium expected utility is

$$\lambda U(1) + (1 - \lambda)U(R)$$

In autarky, consumption is $(y, y + R(1 - y))$ for $0 \leq y \leq 1$. The market allocation $(c_1, c_2) = (1, R)$ dominates every feasible autarkic allocation.
Figure 1: Market allocation and feasible set under autarky
The efficient solution

- The market solution is inefficient because it does not allow trades contingent on the agent’s type, i.e., insurance against liquidity shocks. We represent the efficient allocation as the solution to a planner’s problem.

- Feasibility:

  \[ x + y = 1 \]  \hspace{1cm} (1)
  \[ \lambda c_1 \leq y \]  \hspace{1cm} (2)
  \[ \lambda c_1 + (1 - \lambda) c_2 \leq Rx + y \]  \hspace{1cm} (3)

- The planner’s objective is to choose the investment portfolio \((x, y)\) and the consumption allocation \((c_1, c_2)\) to maximize the typical investor’s expected utility

  \[ \lambda U(c_1) + (1 - \lambda) U(c_2), \]

subject to the various feasibility conditions (1) – (3)
Solving the planner’s problem

- W.l.o.g., we can assume the short asset is used to provide consumption at date 1 and the long asset is used to provide consumption at date 2:

\[ c_1 = \frac{y}{\lambda}; \]
\[ c_2 = \frac{Rx}{1 - \lambda} \]

- Substituting these expressions for consumption into the objective function the planner’s problem is

\[
\max_{0 \leq y \leq 1} \left\{ \lambda U \left( \frac{y}{\lambda} \right) + (1 - \lambda) U \left( \frac{R(1 - y)}{1 - \lambda} \right) \right\}
\]

- A necessary condition for an interior optimum is

\[
U'(c_1) = RU'(c_2) R
\]
The ineffectiveness of the market solution

- The feasible allocations for the planner’s problem are defined by the equation

\[(c_1, c_2) = \left( \frac{y}{\lambda}, \frac{R(1-y)}{(1-\lambda)} \right)\]

- The market allocation corresponds to putting \( y = \lambda \):

\[(c_1, c_2) = \left( \frac{y}{\lambda}, \frac{R(1-y)}{(1-\lambda)} \right) = (1, R)\]

- The first-order condition for optimality (5) at this point is

\[U'(1) = U'(R)R\]

This will be satisfied in the special case of log utility function, but not in general.

- Suppose that

\[U(c) = \frac{1}{1-\sigma}c^{1-\sigma}\]

Then \( \sigma > 1 \) implies that \((c_1, c_2)\) satisfies \( c_1 > 1 \) and \( c_2 < R \) and, conversely, \( \sigma < 1 \) implies that \( c_1 < 1 \) and \( c_2 > R \).
Figure 2: Inefficiency of the market solution
Complete markets

- A consumer can purchase date-\( t \) goods at a price \( q_t \) at date 0, for delivery conditional on being type \( t = 1, 2 \). The consumer’s budget constraint is

\[
q_1 \lambda c_1 + q_2 p (1 - \lambda) c_2 \leq 1, \tag{6}
\]

where \( p = \frac{P}{R} \) is price of date-2 goods in terms of date-1 goods.

- The individual chooses \((c_1, c_2)\) to maximize \( \lambda U(c_1) + (1 - \lambda) U(c_2) \) subject to (6) and the solution must satisfy the first-order conditions

\[
\lambda U'(c_1) = \mu q_1 \lambda \\
(1 - \lambda) U'(c_2) = \mu q_2 p (1 - \lambda).
\]

Then

\[
\frac{U'(C_1)}{U'(C_2)} = \frac{q_1}{q_2 p}.
\]
No-arbitrage conditions

- There are zero profits from investing in the short asset if and only if
  \[ q_1 = 1 \]

- Similarly, there are zero profits from investing in the long asset if and only if
  \[ pq_2 = \frac{1}{R} \]

- These no-arbitrage conditions imply that
  \[ \frac{U'(C_1)}{U'(C_2)} = R, \]
  the condition required for efficient risk sharing
The banking solution

- A bank takes a deposit of one unit of the good, invests it in a portfolio \((x, y)\) and offers the consumer a deposit contract \((c_1, c_2)\)
- Free entry into the banking sector and competition force banks to maximize the ex ante expected utility of the typical depositor subject to a zero-profit constraint
- The bank chooses the investment portfolio \((x, y)\) and the consumption allocation \((c_1, c_2)\) to maximize

\[
\lambda U(c_1) + (1 - \lambda) U(c_2),
\]

subject to the feasibility conditions \((1) - (3)\)
- The incentive-compatibility constraint is automatically satisfied because \(c_1 \leq c_2\)
Financial fragility (multiple equilibria)

- **Liquidation technology:** premature liquidation of the long asset yields $r \leq 1$ units of the good
- If all depositors withdraw at date 1, the liquidated value of bank assets is
  \[ rx + y \leq x + y = 1 \]
- If $c_1 > rx + y$, the bank is insolvent and will be able to pay only a fraction of the promised amount $c_1$ and nothing will be left at date 2

\[
\begin{array}{|c|c|}
\hline
\text{Run} & \text{No Run} \\
\hline
(rx + y, rx + y) & (c_1, c_2) \\
(0, rx + y) & (c_2, c_2) \\
\hline
\end{array}
\]

- It is clear that if
  \[ 0 < rx + y < c_1 < c_2 \]
  then (Run, Run) is an equilibrium and (No Run, No Run) is also an equilibrium
- Suspension of convertibility and the sequential service constraint
Equilibrium bank runs

The Diamond-Dybvig argument shows the possibility of an *unexpected* run

- A run cannot be predicted with certainty
- So, the best we can hope for is a bank run that occurs *with positive probability*

- A **sunspot** (random variable) takes two values, high and low, with probabilities $\pi$ and $1 - \pi$, respectively. Depositors run on the bank when the sunspot is “high” and not when it is “low”

- The bank chooses a portfolio $(x, y)$ and a deposit contract $(c_1, c_2)$, in the expectation that $(c_1, c_2)$ is achieved only if the bank is solvent. In the event of a bank run, the typical depositor will receive the value of the liquidated portfolio $rx + y$ at date 1.

$$
\left( \tilde{C}_1, \tilde{C}_2 \right) = \begin{cases} 
(rx + y, rx + y) & \text{if } S = \text{“high”} \\
(c_1, c_2) & \text{if } S = \text{“low”}
\end{cases}
$$
Figure 3: Runs with positive probability
The optimal portfolio

- The expected utility of the representative depositor can be written

\[
\pi U(y + rx) + (1 - \pi) \{\lambda U(c_1) + (1 - \lambda) U(c_2)\}.
\]

- The optimal portfolio must satisfy the first-order condition

\[
\pi U'(y + rx)(1 - r) + (1 - \pi) U'(c_1) = (1 - \pi) U'(c_2) R.
\]

- If \(\pi = 0\) then this reduces to the familiar condition

\[U'(c_1) = U'(c_2) R.\]

- The possibility of a run \((\pi > 0)\) increases the value of a marginal increase in \(y\) and hence increases the amount of the short asset held in the portfolio.
Figure 4: The optimal portfolio when runs are possible
The optimal deposit contract

- The bank chooses the deposit contract \((c_1^*, c_2^*)\) to satisfy the first-order condition
  \[ U'(c_1^*) = RU'(c_2^*). \]  
  \( (7) \)
- Suppose for simplicity that \( r = 1 \) and relative risk aversion is greater than one. These conditions imply the possibility of a run.
- The long asset now dominates the short asset so, without essential loss of generality, we can assume \( y = 0 \) and \( x = 1 \).
- The deposit contract must solve the decision problem
  \[
  \max \lambda U(c_1) + (1 - \lambda) U(c_2)
  \]
  \[
  \text{s.t. } R\lambda c_1 + (1 - \lambda) c_2 \leq R
  \]
Equilibrium without runs

- The bank can prevent a run by choosing a sufficiently “safe” contract:
  \[ c_1 \leq 1 \]

- If we solve the problem
  \[
  \begin{align*}
  \max & \quad \lambda U(c_1) + (1 - \lambda) U(c_2) \\
  \text{s.t.} & \quad R (\lambda c_1) + (1 - \lambda) c_1 \leq R \\
  & \quad c_1 \leq 1
  \end{align*}
  \]

  we find the solution \((c_1^{**}, c_2^{**}) = (1, R)\)
A characterization of regimes with and without runs

- If the bank anticipates a sunspot with probability $\pi$, it will be better to avoid runs if

$$ \pi U(1) + (1 - \pi) \{ \lambda U(c_1^*) + (1 - \lambda) U(c_2^*) \} > \lambda U(1) + (1 - \lambda) U(R) $$

- The expected utility from the safe strategy $\lambda U(1) + (1 - \lambda) U(R)$ lies between two values:

$$ U(1) < \lambda U(1) + (1 - \lambda) U(R) $$

$$ < \lambda U(c_1^*) + (1 - \lambda) U(c_2^*) $$

- There exists a unique value $0 < \pi_0 < 1$ such that

$$ \pi_0 U(1) + (1 - \pi_0) \{ \lambda U(c_1^*) + (1 - \lambda) U(c_2^*) \} = \lambda U(1) + (1 - \lambda) U(R) $$

- The bank will prefer runs if and only if $\pi < \pi_0$
Figure 5: Equilibrium with positive probability of runs
Essential bank runs

- The long asset has a random return $\tilde{R}$ at date 2 given by

$$
\tilde{R} = \begin{cases} 
R_H & \text{w. pr. } \pi_H \\
R_L & \text{w. pr. } \pi_L 
\end{cases}
$$

If the long asset is prematurely liquidated, it yields $r$ units of the good at date 1. We assume that $$R_H > R_L > r > 0$$

- Without loss of generality we put $c_2 = \infty$ and characterize the deposit contract by $c_1 = d$

- We consider only essential bank runs, that is, runs that cannot be avoided

- At date 1, the budget constraint requires $\lambda d \leq y$. If there is no run, late consumers receive

$$c_{2s} = \frac{R_s(1 - y) + y - \lambda d}{1 - \lambda}$$
Bankruptcy

- The *incentive constraint* requires

\[ d \leq R_s(1 - y) + y \]

- The necessary and sufficient condition for an *essential* bank run is that the incentive constraint is violated, that is,

\[ d > R_s(1 - y) + y \]

- There are three cases to consider:
  - the incentive constraint is never binding and bankruptcy never occurs;
  - bankruptcy is a possibility but the bank finds it optimal to choose a deposit contract and portfolio so that the incentive constraint is (just) satisfied;
  - the costs of distorting the choice of deposit contract and portfolio are so great that the bank finds it optimal to allow bankruptcy in the low state
Case I: The incentive constraint is not binding in equilibrium

- We solve the intermediary’s decision problem without the incentive constraint and then check whether the constraint is binding or not.

- The intermediary chooses $y$ and $d$ to maximize expected utility, assuming that there is no bank run.
  
  ▶ With probability $\lambda$, the depositor is an early consumer and receives $d$ regardless of the state.
  
  ▶ With probability $1 - \lambda$, the depositor is a late consumer and his consumption in state $s$ is $R_s(1 - y) + y - \lambda d$ divided by the number of late consumers $1 - \lambda$.

- Thus, the expected utility

$$
\lambda U(d) + (1 - \lambda) \left\{ \pi_H U \left( \frac{R_H(1 - y) + y - \lambda d}{1 - \lambda} \right) + \pi_L U \left( \frac{R_L(1 - y) + y - \lambda d}{1 - \lambda} \right) \right\}
$$

is maximized subject to $0 \leq y \leq 1$ and $\lambda d \leq y$.
The optimal portfolio

Assuming an interior solution, i.e., $0 < y < 1$, the optimal $(y, d)$ is characterized by the necessary and sufficient first-order conditions:

$$U'(d) = \left\{ \pi_H U' \left( \frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) + \pi_L U' \left( \frac{R_L(1-y) + y - \lambda d}{1-\lambda} \right) \right\} \geq 0,$$

and

$$\pi_H U' \left( \frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) (1 - R_H) + \pi_L U' \left( \frac{R_L(1-y) + y - \lambda d}{1-\lambda} \right) (1 - R_L) \leq 0,$$

with equality in each case if $\lambda d < y$. 

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Figure 6: Consumption as a function of R

![Graph showing consumption as a function of R]
Equilibrium with a non-binding IC constraint

- A solution to these inequalities, \((y^*, d^*)\), represents an equilibrium if

\[
d^* \leq R_L (1 - y^*) + y^*
\]

- Let \(U^*\) denote the maximized value of expected utility corresponding to \((y^*, d^*)\)

- If the low state return \(R_L\) is sufficiently high, say \(R_L = R_L^*\), then the incentive constraint is never binding

- Early consumers receive

\[
c_{1s} = d = \frac{y}{\lambda}
\]

and late consumers receive

\[
c_{2s} = \frac{R_s (1 - y)}{(1 - \lambda)}
\]

in each state \(s = H, L\)
Case II: The incentive constraint is binding in equilibrium

- Suppose that \((y^*, d^*)\) does not satisfy the incentive constraint.
- If the intermediary chooses not to default, the decision problem is to choose \((y, d)\) to maximize

\[
\lambda U(d) + (1 - \lambda) \left\{ \pi_H U(c_H) + \pi_L U(c_L) \right\}
\]

subject to the feasibility constraints

\[
0 \leq y \leq 1 \text{ and } \lambda d \leq y
\]

and the incentive constraints

\[
c_{2s} = \frac{R_s (1 - y) + y - \lambda d}{1 - \lambda} \geq d, \text{ for } s = H, L
\]

- The incentive constraint will only bind in the low state \(s = L\).
Substituting for $c_{2L} = d$, the expression for expected utility can be written as

$$\lambda U(d) + (1 - \lambda) \left\{ \pi_H U \left( \frac{R_H (1 - y) + y - \lambda d}{1 - \lambda} \right) + \pi_L U(d) \right\},$$

where the incentive constraint implies that $d \equiv R_L (1 - y) + y$

In this case, the first-order condition that characterizes the choice of $y$ takes the form

$$\lambda U'(d) (1 - R_L) + (1 - \lambda) \left\{ \pi_H U' \left( \frac{R_H (1 - y) + y - \lambda d}{1 - \lambda} \right) \times \left( \frac{1 - R_H - 1 + \lambda R_L}{1 - \lambda} \right) + \pi_L U'(d) (1 - R_L) \right\} \leq 0,$$

with equality if $\lambda d < y$

Let $(y^{**}, d^{**})$ denote the solution to this problem and let $U^{**}$ denote the corresponding maximized expected utility.
Case III: The incentive constraint is violated in equilibrium

- Default in the low state implies expected utility is

\[
\pi_H \left\{ \lambda U(d) + (1 - \lambda) U \left( \frac{R_H(1 - y) + y - \lambda d}{1 - \lambda} \right) \right\} 
+ \pi_L U (r(1 - y) + y)
\]

- The FOCs for an optimum take the form

\[
\pi_H \left\{ \lambda U'(d) - \lambda U' \left( \frac{R_H(1 - y) + y - \lambda d}{1 - \lambda} \right) \right\} \geq 0,
\]

\[
\pi_H U' \left( \frac{R_H(1 - y) + y - \lambda d}{1 - \lambda} \right) (1 - R_H) + 
\pi_L U' (r(1 - y) + y) (1 - R_L) \leq 0,
\]

with equality if \( \lambda d < y \)

- Let \( (d^{***}, y^{***}) \) and \( U^{***} \) denote the solution and maximum value

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• If \( R_L = R_L^{***} \) then bankruptcy occurs in the low state and both early and late consumers receive the same consumption in the low state:

\[
c_{1L} = c_{2L} = y + R_L (1 - y) < d
\]

• In the high state,

\[
c_{1H} = d\text{ and } c_{2H} = \frac{R_H (1 - y)}{1 - \lambda}
\]

• This is an equilibrium solution only if

\[
d^{***} > R_L (1 - y) + y,
\]

and

\[
U^{***} > U^{**}
\]

• The first condition guarantees that the incentive constraint is violated
• The second condition guarantees that default is preferred to solvency