Monday Lecture 1
Welfare Economics

August 6, 2012
An exchange economy

- There are $\ell$ commodities indexed $h = 1, \ldots, \ell$. The commodity space is $X = \mathbb{R}^\ell$.
- There are $m$ agents (consumers), indexed $i = 1, \ldots, m$, with consumption set $X$, a utility function $U_i : X \to \mathbb{R}$ and initial endowment of commodities $e_i \in X$.
- The $m$-tuple $\mathcal{E} = \{(U_i, e_i)\}_{i=1}^m$ is called an exchange economy.
- An allocation is an array $x = (x_1, \ldots, x_m) \in X^m$.
- An allocation $x \in X^m$ is attainable if

$$\sum_{i=1}^m x_i = \sum_{i=1}^m e_i.$$
Walrasian equilibrium

- A **Walrasian equilibrium** consists of an attainable allocation $x^*$ and a price vector $p^* \neq 0$ such that, for every $i = 1, \ldots, m$, $x_i^*$ maximizes $U_i$ on the budget set

$$B_i(p^*) = \{ x_i \in X_i : p^* \cdot x_i \leq 0 \}.$$ 

- We call $x^*$ a **Walras allocation** if $(x^*, p^*)$ is a Walrasian equilibrium, for some price vector $p^* \neq 0$.

- An attainable allocation $x$ is said to be **weakly Pareto efficient** if there does not exist an attainable allocation $y$ such that $U_i(y_i) > U_i(x_i)$, for every $i = 1, \ldots, m$.

- An attainable allocation $x$ is said to be **(strongly) Pareto efficient** if there does not exist an attainable allocation $y$ such that $U_i(y_i) \geq U_i(x_i)$, for every $i = 1, \ldots, m$, and $U_i(y_i) > U_i(x_i)$ for at least one $i$. 
First Theorem of Welfare Economics

Theorem (First Theorem of Welfare Economics)

A Walras allocation is weakly Pareto efficient.

Proof.

Let \((\mathbf{x}, \mathbf{p})\) be an equilibrium and suppose, contrary to what we want to prove, that \(\mathbf{x}'\) is attainable and is strictly preferred to \(\mathbf{x}\) by every agent \(i\). Then \(\mathbf{p} \cdot \mathbf{x}'_i > \mathbf{p} \cdot \mathbf{e}_i\) for every \(i\), so \(\sum_i \mathbf{p} \cdot \mathbf{x}'_i > \sum_i \mathbf{p} \cdot \mathbf{e}_i\), contradicting \(\sum_i \mathbf{x}'_i = \sum_i \mathbf{e}_i\).

Example

As an example of an equilibrium that is weakly but not strongly Pareto efficient, use the Edgeworth Box with “thick” indifference curves.
Agent $i$ is **locally non-satiable** if, for any point $x_i$ in the consumption set $X_i$ and any $\varepsilon > 0$ there is a consumption bundle $x'_i \in X_i$ such that $\|x'_i - x_i\| < \varepsilon$, $x'_i \gg x_i$, and $U_i(x'_i) > U_i(x_i)$.

**Theorem (First Theorem of Welfare Economics)**

A Walras allocation $x$ is strongly Pareto efficient if every agent is locally non-satiable.

**Proof.**

Let $(x, p)$ be an equilibrium and suppose, contrary to what we want to prove, that $x$ is attainable and is weakly preferred to $x$ by every agent $i$ and strictly preferred by some agent $i$. Local non-satiability implies that $p \cdot x_i \geq p \cdot e_i$ for every $i$ and the inequality is strict for some $i$, so $\sum_i p \cdot x_i > \sum_i p \cdot e_i$, contradicting $\sum_i x_i = \sum_i e_i$.  

\[\square\]
Second Theorem of Welfare Economics

- An attainable allocation $x$ can be **decentralized** if there exists a price vector $p \neq 0$ such that, for every $i = 1, \ldots, m$,

  $$U_i(y_i) > U_i(x_i) \implies p \cdot y_i > p \cdot x_i.$$ 

- Let $P_i(x_i)$ denote the set of points that is preferred to $x_i$ by agent $i$, that is,

  $$P_i(x_i) = \{ y_i \in X : U_i(y_i) > U_i(x_i) \},$$

  for every $i = 1, \ldots, m$.

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**Theorem (Second Theorem of Welfare Economics)**

Suppose that $x^*$ is a weakly efficient allocation and suppose that $U_i$ is continuous and l.n.s. and $P_i(x)$ is convex for every $i = 1, \ldots, m$. Then $x^*$ can be decentralized using a price vector $p^* \neq 0$. 
Proof of Second Theorem

- Let $Z_i = P_i(x_i^*) - \{x_i^*\}$ for each $i$ and let $Z = \sum_i Z_i$. Then
  
  (a) $Z$ is nonempty because $U_i$ is l.n.s.,
  
  (b) $Z_i$ is open because $U_i$ is continuous, and
  
  (c) $Z_i$ is convex because $P_i(x_i^*)$ is convex.

- We claim that $0 \notin Z$ because $x^*$ is weakly Pareto efficient. If not, there exist vectors $z_i \in Z_i$ such that $\sum_i z_i = 0$. Then $y = x^* + z$ is an attainable allocation and $U_i(y_i) > U_i(x_i^*)$ for every $i$, a contradiction.

**Theorem (Minkowski lemma)**

Let $S$ be a nonempty, open and convex set and suppose $0 \notin S$. Then there exists a vector $p \in R^\ell$ such that $p \neq 0$ and $p \cdot x > 0$ for any $x \in S$. 

Proof of Second Theorem continued

- By the Minkowski lemma, there exists a vector $p^* \neq 0$ such that $p^* \cdot z > 0$ for any $z \in Z$.
- The continuity of $U_i$ implies that $0 \in \overline{Z}_i$ for all $i$.
- This inequality implies that $p^* \cdot z_i > 0$ for any $z_i \in Z_i$, as required.
- This shows that $x^*$ is decentralized using the price vector $p^*$. 

An economy with risk

- Assume a finite number of states of nature, $s = 1, ..., S$, with common probability distribution $\pi(s)$.
- Agent $i$ has a vNM utility function $U_i : X \rightarrow \mathbb{R}$.
- The commodity space is $X^S = \{x : S \rightarrow X\}$, where $x(s)$ denotes the bundle of contingent commodities delivered in state $s$. The endowment of agent $i$ is $e_i \in X^S$.
- An exchange economy is defined by the $m$-tuple $E = \{(U_i, e_i)\}_{i=1}^m$.
- An allocation for the economy is now an $m$-tuple $x = \{x_i\}_{i=1}^m$ such that $x_i \in X^S$ for each $i$. The allocation $x$ is attainable if $\sum_{i=1}^m x_i = \sum_{i=1}^m e_i$.
- A Walrasian equilibrium consists of an attainable allocation $x^*$ and a price vector $p^* \in X^S$ such that, for every $i = 1, ..., m$, the consumption bundle $x_i^*$ maximizes $\sum_{s=1}^S \pi(s) U_i(x_i(s))$ in the budget set

$$B_i(p^*) = \left\{ y_i \in X^S : p^* \cdot y_i \leq p^* \cdot e_i \right\}.$$
Arrow securities

- Securities are traded before the true state of nature is revealed.
- The $\ell$ goods are traded on spot markets after the true state is revealed.
- An **Arrow security** promises one unit of account in some state $s$ and nothing in other states.
- We assume that there is a complete set of Arrow securities, one for each state $s$.
- Let $z_{is}$ denote agent $i$’s demand (positive or negative) for Arrow security $s$ and let
  \[
  z_i = (z_{i1}, ..., z_{is}, ..., z_{iS})
  \]
denote the vector of security demands for agent $i$. The set of possible security demands is denoted by $Z \equiv \mathbb{R}^S$. 

Allocations

- A security allocation is an $m$-tuple $\mathbf{z} = (z_1, \ldots, z_i, \ldots, z_m) \in \mathbb{Z}^m$. An allocation of contingent commodities is an $m$-tuple $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_m) \in (\mathcal{X}^S)^m$, where $x_i$ is the vector of demands for contingent commodities for agent $i$.

- An allocation consists of an order pair $(\mathbf{x}, \mathbf{z}) \in (\mathcal{X}^S)^m \times \mathbb{Z}^m$, where $\mathbf{x}$ is an allocation of contingent commodities and $\mathbf{z}$ is the allocation of securities.

- An allocation $(\mathbf{x}, \mathbf{z})$ is attainable if

$$\sum_{i=1}^m (x_i, z_i) = \sum_{i=1}^m (e_i, 0).$$

- Let let $\mathbf{q} = (q_1, \ldots, q_s, \ldots, q_S)$ $q_s$ denote the vector of Arrow security prices, let $p(s) \neq 0$ denote the vector of commodity prices in state $s$ and let $\mathbf{p} = (p(1), \ldots, p(s), \ldots p(S))$. 
Equilibrium with Arrow securities

Definition

An **equilibrium with Arrow securities** consists of an attainable allocation \((x^*, z^*)\) and a **price system** \((p^*, q^*)\) such that, for each agent \(i\), the ordered pair \((x_i^*, z_i^*)\) maximizes

\[
\sum_{s=1}^{S} \pi(s) U_i(x_i(s))
\]

subject to the budget constraints

\[
q^* \cdot z_i \leq 0
\]

and

\[
p^*(s) \cdot x_i(s) \leq p^*(s) \cdot e_i(s) + z_{is}, \ \forall s.
\]
If \((x^*, p^*)\) is a Walrasian equilibrium, then \((x^*, z^*, p^*, q^*)\) is an equilibrium with Arrow securities, where \(q^* = (1, \ldots, 1)\) and \(z_i^*\) is defined by

\[z_{is}^* = p^*(s) \cdot (x_i^*(s) - e_i(s)), \forall s.\]

Conversely, if \((x^*, z^*, p^*, q^*)\) is an equilibrium with Arrow securities, then \((x^*, p)\) is an equilibrium with complete markets, where \(p\) is defined by

\[p(s) = q_s p^*(s), \forall s.\]
Proof of Arrow’s Theorem

Suppose that \((x^*, p^*)\) is a Walrasian equilibrium and let \(z^*\) and \(q^*\) be defined by

\[ q_s = 1 \]

and

\[ z_{is}^* = p^*(s) \cdot (x_i^*(s) - e_i(s)), \]

for any \(s = 1, \ldots, S\) and \(i = 1, \ldots, m\). We want to show that \((x^*, z^*, p^*, q^*)\) is an equilibrium with Arrow securities.

**Step 1:** Note that, for every state \(s\),

\[ \sum_{i=1}^{m} z_{is}^* = \sum_{i=1}^{m} p^*(s) \cdot (x_i^*(s) - e_i(s)) = 0 \]

because \(x^*\) is attainable, so \(\sum_{i=1}^{m} z_i^* = 0\) and, hence, \((x^*, z^*)\) is attainable.
Proof continued

**Step 2:** For every agent $i$,

\[ q^* \cdot z_i^* = \sum_{s=1}^{S} p^* (s) \cdot (x_i^*(s) - e_i(s)) \leq 0, \]

since $x_i^* \in B_i(p^*)$. Also, for every $i$,

\[ p^* (s) \cdot x_i^*(s) \leq p^* (s) \cdot e_i(s) + z_is^*, \quad \forall s. \]

Hence, $(x_i^*, z_i^*)$ belongs to the budget set of agent $i$.

**Step 3:** Now suppose that $(x_i, z_i)$ belongs to the budget set of agent $i$. The budget constraints imply that

\[ \sum_{s=1}^{S} p^* (s) \cdot (x_i(s) - e_i(s)) \leq \sum_{s=1}^{S} z_is \leq 0, \]

so $x_i$ belongs to the Walrasian budget set $B_i(p^*)$. Since $x_i^*$ maximizes expected utility over the budget set $B_i(p^*)$, $(x_i^*, z_i^*)$ must maximize expected utility over the budget set in the equilibrium with Arrow securities.
Proof continued

**Step 4:** Suppose that \((x^*, z^*, p^*, q^*)\) is an equilibrium with Arrow securities. Define a price vector \(\mathbf{p}\) for a Walrasian equilibrium by putting

\[
\mathbf{p}(s) = q^*_s \mathbf{p}^*(s), \quad \forall s.
\]

Clearly, \(x^*\) is attainable because \((x^*, z^*)\) is attainable. Also, \(x^*_i\) belongs to the budget set \(B_i(\mathbf{p})\) because

\[
\mathbf{p} \cdot (x^*_i - e_i) = \sum_{s=1}^{S} \mathbf{p}(s) \cdot (x^*_i(s) - e_i(s)) \\
= \sum_{s=1}^{S} q^*_s \mathbf{p}^*(s) \cdot (x^*_i(s) - e_i(s)) \\
\leq \sum_{s=1}^{S} q^*_s z^*_i \leq 0
\]

from the budget constraints of the equilibrium with Arrow securities.
Proof continued

Step 5: Now suppose that $x_i$ belongs to $B_i(p)$. Define $z_i$ by putting

$$z_{is} = p^*(s) \cdot (x_i(s) - e_i(s)), \quad \forall s.$$ 

Then

$$q^* \cdot z_i = \sum_{s=1}^{S} q_s p^*(s) \cdot (x_i^*(s) - e_i(s))$$

$$= \sum_{s=1}^{S} p(s) \cdot (x_i^*(s) - e_i(s))$$

$$= p \cdot (x_i^* - e_i) \leq 0,$$

so $(x_i, z_i)$ belongs to the budget set of the equilibrium with Arrow securities. Since $(x_i^*, z_i^*)$ maximizes expected utility in the budget of the equilibrium with Arrow securities, agent $i$ must prefer $x_i^*$ to $x_i$. Thus, $x_i^*$ is optimal in the budget set $B_i(p)$. This completes the proof that $(x^*, p)$ is a Walrasian equilibrium.
Necessary conditions for optimal risk sharing

- If $U_i(x_{is})$ is $C^1$, a necessary first-order condition for optimality is that

$$\pi_s U'_i(x^*_{is}) = \lambda_i p^*_s$$

for every state $s = 1, \ldots, S$.

- Eliminating $\lambda_i$ we get

$$\frac{\pi_s U'_i(x^*_{is})}{\pi_{s'} U'_i(x^*_{is'})} = \frac{\lambda_i p^*_s}{\lambda_i p^*_{s'}} = \frac{p^*_s}{p^*_{s'}}$$

for any states $s, s'$.

- This immediately implies that, for any pair of agents $i$ and $j$,

$$\frac{\pi_s U'_i(x^*_{is})}{\pi_{s'} U'_i(x^*_{is'})} = \frac{\pi_s U'_j(x^*_{js})}{\pi_{s'} U'_j(x^*_{js'})}.$$
The Borch conditions

- Canceling probabilities on both sides, we have the **Borch conditions**:

\[
\frac{U'_i(x^*_i)}{U'_i(x^*_{i'})} = \frac{U'_j(x^*_j)}{U'_j(x^*_{j'})}.
\]

- The Borch conditions are the necessary and sufficient conditions for efficient risk sharing when the utility functions \(u_i(x_i)\) are concave and continuously differentiable.

- They imply the necessity of **coinsurance**, that is, every agent’s consumptions moves in the same direction between states.

- If utility functions are strictly concave, coinsurance has a striking implication: efficient risk sharing implies that each agent’s consumption is a function of the total endowment.
Demand functions

Let $\mathcal{E} = \{(X_i, e_i, U_i)\}$ be an exchange economy satisfying the following properties:

- $X_i = \mathbb{R}^l_+$ and $e_i \geq 0$ for any $i = 1, ..., m$.
- $U_i : \mathbb{R}^l_+ \to \mathbb{R}$ is increasing, continuous and strictly quasi-concave.
- Let $P = \{p \in \mathbb{R}^l_+ : p_h \gg 0, p_\ell = 1\}$ and let $\bar{P} = \text{cl} P$ denote the closure of $P$. For any $p \in P$, let $B_i(p, p \cdot e_i) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$ and let

$$\xi_i(p) = \arg \max \{U_i(x_i) : x_i \in B_i(p, p \cdot e_i)\}.$$ 

- $\xi_i(p)$ is a singleton for any $p \in P$ and $\xi_i : P \to \mathbb{R}^l_+$ is a well-defined function. Moreover, $\xi_i$ is continuous on $P$ and, for any sequence $\{p^q\}$ in $P$ converging to $p^0 \in \bar{P}\setminus P$, $\|\xi_i(p^q)\| \to \infty$. 

Excess demand functions

- Define the individual excess demand function $z_i : P \rightarrow \mathbb{R}^l$ for agent $i$ by putting
  
  $$z_i(p) = \xi_i(p) - e_i,$$
  
  for every $p \in P$ and define the aggregate excess demand function for $E$, denoted by $z : P \rightarrow \mathbb{R}^l$, by putting
  
  $$z(p) = \sum_{i=1}^{m} z_i(p),$$
  
  for every $p \in P$.

- Under the maintained assumptions, $z : P \rightarrow \mathbb{R}^l$ is well defined for the pure exchange economy $E$. The function $z$ is continuous and satisfies the boundary condition
  
  $$\|z(p^q)\| \rightarrow \infty$$
  
  for any sequence $\{p^q\}$ in $P$ such that $p^q \rightarrow p^0 \in \bar{P} \setminus P$. 

Regular economies

Let \( U \subset \mathbb{R}^n \) be an open set, let \( f : U \rightarrow \mathbb{R}^n \) be a function, and suppose that \( \mathbf{x} \in U \) is a solution of the equation

\[
    f(\mathbf{x}) = 0.
\]

Then we say that \( \mathbf{x} \) is **locally unique** (or a locally unique solution) if there is some open set \( V \subset U \) such that \( \mathbf{x} \in V \) and there does not exist \( \mathbf{y} \neq \mathbf{x} \) in \( V \) such that \( f(\mathbf{y}) = 0 \). The following theorem is often used to establish local uniqueness.

**Theorem**

**Inverse Function Theorem.** Let \( U \subset \mathbb{R}^n \) be open and \( f : U \rightarrow \mathbb{R}^n \) be \( C^r \), \( 1 \leq r \leq \infty \), at \( \mathbf{x} \). If the matrix of derivatives \( \nabla f(\mathbf{x}) \) is nonsingular (invertible), then there is an open set \( V \subset \mathbb{R}^n \) such that \( f(\mathbf{x}) \in V \) and a \( C^r \) function \( f^{-1} : V \rightarrow \mathbb{R}^n \) such that \( f^{-1}(f(\mathbf{y})) = \mathbf{y} \) on a neighborhood of \( \mathbf{x} \). Moreover,

\[
    \nabla f^{-1}(f(\mathbf{x})) = [\nabla f(\mathbf{x})]^{-1}.
\]

A \( C^1 \) inverse at \( f(\mathbf{x}) \) can exist only if \( \nabla f(\mathbf{x}) \) is nonsingular.
Regularity

We assume that $z : P \to \mathbb{R}^\ell$ is a member of the class $C^r$ for $1 \leq r \leq \infty$, normalize the price vector by putting $p_\ell = 1$ and denote the vector of excess demands of the first $\ell - 1$ goods by

$$\hat{z}(p) = (z_1(p), \ldots, z_{\ell-1}(p)).$$

The excess demand function $z$ satisfies Walras’ law, that is,

$$p \cdot z(p) = 0,$$

for any $p \in P$. If all but one market clears, the remaining market must clear also.

A normalized price vector $p = (p_1, \ldots, p_{\ell-1}, 1)$ constitutes a Walrasian equilibrium if and only if it solves the system of $\ell - 1$ equations in $\ell - 1$ unknowns

$$\hat{z}(p) = 0.$$
Local uniqueness

**Definition**

An equilibrium price vector \( p = (p_1, ..., p_{\ell-1}, 1) \) is **regular** if the \((\ell - 1) \times (\ell - 1)\) matrix of price effects \( \nabla z(p) \) is non-singular, that is, has rank \( \ell - 1 \). If every normalized equilibrium price vector is regular we say that the economy is **regular**.

**Theorem**

Any regular (normalized) equilibrium price vector \( p = (p_1, ..., p_{\ell-1}, 1) \) is **locally unique**. That is, for some \( \varepsilon > 0 \) and any \( p' \neq p \) such that \( p'_\ell = p_\ell = 1 \) and \( \|p' - p\| < \varepsilon \), \( z(p') \neq 0 \). Moreover, if the economy is regular, then the number of normalized equilibrium price vectors is finite.
Genericity I

A property is said to be generic if it holds for all parameters outside a negligible set, for example, a set of measure zero. We want to show that regularity is such a generic property.

**Theorem (Transversality)**

Suppose that \( f: \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n \) if the \( n \times (m + p) \) matrix \( \nabla f(x; q) \) has rank \( n \) whenever \( f(x, q) = 0 \) then for almost every \( q \), the \( n \times m \) matrix \( \nabla_x f(x; q) \) has rank \( n \) whenever \( f(x; q) = 0 \).

Write the excess demand function \( \hat{z}(p; e) \) to show the dependence on the vector of endowments \( e = (e_1, ..., e_m) \).

**Theorem (Rank Condition)**

For any \( p \) and \( e \), the rank of \( \nabla_e \hat{z}(p; e) \) is \( \ell - 1 \).
The next proposition follows directly from the Transversality Theorem and the Rank Condition.

**Theorem**

*For almost every vector of initial endowments $\mathbf{e} = (e_1, \ldots, e_m)$, the economy defined by $\{(u_i, e_i)\}_{i=1}^m$ is regular.*
Asset economies

- There two dates $t = 0, 1$ and a finite number of states $s = 0, 1, \ldots, S$. The state of nature is unknown at date 0 but has a common probability distribution $\pi(s)$; the true state is revealed at date 1.

- There are $\ell + 1$ goods, indexed by $h = 0, \ldots, \ell$ so the commodity space is $\mathbb{R}^{(\ell+1)(S+1)}$.

- There are $m + 1$ economic agents, indexed by $i = 0, \ldots, m$, characterized by the consumption set $\mathbb{R}^{(\ell+1)(S+1)}_+$, an endowment $e_i \in \mathbb{R}^{(\ell+1)(S+1)}_+$ and a utility function $U_i : \mathbb{R}^{(\ell+1)(S+1)}_+ \rightarrow \mathbb{R}$.

- There is a finite set of assets, indexed by $k = 0, 1, \ldots, K$, all in zero net supply.

- Asset $k$ is defined by a payoff vector $a_k = (a_{k0}, \ldots, a_{KS})$, where $a_{ks}$ is the return in terms of the numeraire good $h = 0$ in state $s$.

- The payoff matrix $A = [a_{ks}]_{(K+1) \times (S+1)}$ characterizes the possibilities of trade between periods and states.
Allocations and prices

- An **allocation** for the economy is an array \((x, z) = \{(x_i, z_i)\}_{i=1}^m\) such that \(x_i \in \mathbb{R}_{(+)}^{(\ell+1)(S+1)}\) and \(z_i \in \mathbb{R}^K\) for each \(i = 1, \ldots, m\).

- An allocation \((x, z)\) is **attainable** if

  \[
  \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \quad \text{and} \quad \sum_{i=1}^m z_i = 0.
  \]

- A **price system** consists of a pair of price vectors \((p, q)\), where \(p \in \mathbb{R}^{(\ell+1)(S+1)}\) and \(q \in \mathbb{R}^K\).

- Assume free disposal, so that \((p, q) \succeq (0, 0)\), w.l.o.g.

- Partition the consumption bundle \(x_i\) for agent \(i\) into the sub-bundles \(x_i(s)\) at states \(s = 0, 1, \ldots, S\), where \(s = 0\) denotes the first date, and write \(x_i = (x_i(0), x_i(1), \ldots, x_i(S))\).

- Similarly, partition the price system \(p\) into the sub-vectors \(p(s)\), for \(s = 0, 1, \ldots, S\), and write \(p = (p(0), \ldots, p(S))\).
An **equilibrium** for the economy consists of an attainable allocation \((x^*, z^*)\) and a price system \((p, q)\) such that, for every agent \(i = 1, \ldots, m\),

\[(x^*_i, z^*_i) \text{ maximizes } U_i(x_i) \text{ subject to the constraints}

\[
q \cdot z_i \leq 0
\]

\[
p(s) \cdot x_i(s) \leq p(s) \cdot e_i(s) + \sum_{k=1}^{K} z_k a_{ks},
\]

for \(s = 0, 1, \ldots, S\).
Assumptions

(A.1) $U_i$ is continuous and quasi-concave on $\mathbb{R}_+^{(\ell+1)(S+1)}$ and the range of $U_i$ can be extended to $\mathbb{R} \cup \{-\infty\}$.

(A.2) $e_i \gg 0$.

(A.3) $U_i$ is increasing in the numeraire good at every state $s = 1, \ldots, S$ at date 1.

(A.4) Free assets give rise to arbitrage: there exists a portfolio $\mathbf{z} \in \mathbb{R}^{K+1}$ such that $\mathbf{z}^t \mathbf{A} \succ 0$.

(D.1) $\mathbf{A}$ has full row rank.

(D.2) $U_i$ is $C^2$, $DU_i \gg 0$ and $D^2 U_i$ is negative definite on $\mathbb{R}_+^{(\ell+1)(S+1)}$.

(D.3) The closure of the indifference curves of $U_i$ do not intersect the boundary of $\mathbb{R}_+^{(\ell+1)(S+1)}$.

(S) The asset market is incomplete: $K < S$.

(CS) Every set of $K + 1$ columns of $\mathbf{A}$ are linearly independent and there exists a portfolio $\mathbf{z}$ such that $\mathbf{a}(s) \cdot \mathbf{z} \neq 0$, for all states $s = 1, \ldots, S$. 
Spot market equilibrium relative to $z^*$

**Theorem**

An equilibrium exists if (A.1) through (A.4) are satisfied.

**Definition**

Let $z$ be a fixed but arbitrary profile of assets satisfying

$$
\sum_{i=1}^{m} z_i = 0
$$

and define a **spot market equilibrium relative to $z$** to be an attainable allocation $(\hat{x}, \hat{z})$ and a price system $(\hat{p}, \hat{q})$ such that, for every agent $i = 1, \ldots, m$, $\hat{x}_i$ maximizes $U_i(x_i)$ subject to the constraints

$$
\hat{q} \cdot z_i \leq 0
$$

$$
\hat{p}(s) \cdot x_i(s) \leq \hat{p}(s) \cdot e_i(s) + \sum_{k=1}^{K} z_{ik} a_{ks},
$$
The space of economies

- Let $\mathcal{E} \subset \mathbb{R}^{(S+1)(\ell+1)(m+1)}_{++}$ be an open set of endowments of each of the $m$ agents and assume that $\mathcal{E}$ is bounded and that the closure of $\mathcal{E}$ does not intersect the boundary of $\mathbb{R}^{(S+1)(\ell+1)(m+1)}_{++}$.

- $\mathcal{U}$ is assumed to be a finite dimensional manifold of utility functions satisfying the assumptions previously assumed and sufficiently rich in perturbations so that, if $U_i \in \mathcal{U}$, then $U_i + \epsilon f \in \mathcal{U}$ for $\epsilon > 0$ sufficiently small, where $f$ is any smooth function.

- The space of economies is identified with the parameters in $\mathcal{E} \times \mathcal{U}^m$.

- A set of economies $D \subset \mathcal{E} \times \mathcal{U}^m$ is said to be **generic** if it is an open dense subset of $\mathcal{E} \times \mathcal{U}^m$ with a null complement. (A null set is here interpreted to be a set of measure zero).
Regularity

**Theorem**

If (A1) through (A.4) and (D.1) through (D.3) are satisfied, then for any choice of utilities $U \in \mathcal{U}$, there is a generic set $E(U)$ of endowments in $\mathcal{E}$ such that for every economy $(e, U)$ with $e \in E(U)$, the set of competitive equilibria is a continuously differentiable function of the endowment allocation $e$.

**Theorem**

If (A1) through (A.4) and (D.1) through (D.3) are satisfied, then there is a generic set of economies $D \subset \mathcal{E} \times \mathcal{U}^m$ on which
(i) the set of competitive equilibria is finite and is a continuously differentiable function of the endowment and utility assignment $(e, U)$;
(ii) the spot market competitive equilibrium corresponding to any competitive portfolio allocation is, locally, a continuously differentiable function of the portfolio allocation $z$. 
Constrained inefficiency

- An attainable allocation \((x, z)\) is **Pareto efficient** if there does not exist an attainable allocation \((x', z')\) such that \(U_i(x'_i) \geq U_i(x_i)\) for \(i = 1, \ldots, m\) and the inequality is strict for some \(i\).

**Proposition**
- If the asset market is incomplete \((S)\) and if \((A.1)\) through \((A.4)\) and \((D.1)\) through \((D.3)\) are satisfied, then for any economy \((e, U) \in D\), a generic set, all competitive equilibria are Pareto inefficient.

- An equilibrium allocation \((x^*, z^*)\) is said to be **constrained efficient** if there does not exist a feasible portfolio profile \(\hat{z}\) and spot market equilibrium relative to \(\hat{z}\), say \((\hat{x}, \hat{z}, \hat{p}, \hat{q})\) such that \(U_i(\hat{x}^*_i) \geq U_i(x^*_i)\) for \(i = 1, \ldots, m\) and the inequality is strict for some \(i\).
A generic result

Theorem

Suppose that $0 < 2\ell \leq m < S\ell$. If the asset market is incomplete (S) and if (A.1) through (A.4), (D.1) through (D.3) and (CS) are satisfied, then for any economy $(e, U) \in D$, a generic set all competitive equilibria are constrained inefficient as long as there are at least two assets, $K + 1 \geq 2$. If $K + 1 \geq 3$, this remains true even if the reallocation of assets must satisfy the asset budget constraint for each individual at the equilibrium asset prices.