Friday Lecture 1
Liquidity Hoarding

August 10, 2012
As early as August 2007, European banks reported difficulty borrowing in the interbank market (Acharya and Merrouche, 2009; Heider, Hoerova and Holthausen, 2008; Ashcraft, McAndrews and Skeie, 2009).

Central banks in Europe and US forced to provide liquidity to financial system

Two explanations for interbank “freeze”

- counterparty risk
- liquidity hoarding

These explanations are not unrelated, of course
Model

- A large number of bankers hold two types of assets, liquid assets ("cash") and illiquid assets ("assets")
- Bankers are subject to stochastic liquidity shocks, requiring payment of one unit of cash
- Illiquid bankers sell assets to obtain cash
- If asset prices are too low (cost of cash is too high), bankers may choose to default
- Default results in bankruptcy and liquidation
- Bankers weigh the opportunity cost of holding cash against the risk of costly bankruptcy
Time, goods and assets

- **Time**: Time is divided into four dates, $t = 0, 1, 2, 3$

- **Goods**: There is a single consumption good at each date

- **Assets**: There are two assets, a liquid asset (‘cash’) and an illiquid asset (‘the asset’)

- **Returns**:
  - one unit of cash can be turned into one unit of consumption at any date;
  - one unit of the asset pays a return of $R > 1$ units of cash (consumption) at date 3
Bankers

- **Bankers**: There is a continuum of ex ante identical, risk-neutral bankers, \( i \in [0, 1] \)
- **Endowments**: each banker is endowed with one unit of cash and one unit of the asset at date 0
- **Preferences**: a banker values consumption only at date 0 and date 3

\[
U(c_0, c_3) = \rho c_0 + c_3, \quad (\rho > 1)
\]

- **Activities**:
  - **at date 0**: bankers choose the level of liquidity in their portfolios
  - **at dates 1 and 2**: bankers receive liquidity shocks and trade assets to obtain liquidity
  - **at date 3**: asset returns are realized
Debt and default

- **Liquidity shocks**: A banker receives a liquidity shock at date $t = 1, 2, 3$ with probability

$$\begin{align*}
\theta_1 & \quad \text{if } t = 1 \\
(1 - \theta_1) \theta_2 & \quad \text{if } t = 2 \\
(1 - \theta_1)(1 - \theta_2) & \quad \text{if } t = 3
\end{align*}$$

where $\theta_1 \sim F_1(\theta_1)$, $\theta_2 \sim F_2(\theta_2)$, and $\theta_1$ and $\theta_2$ have support $[0, 1]$.

- **Default**: On receiving a shock, a banker must either pay one unit of cash to discharge a senior claim or default and suffer a loss of 100% of the value of his portfolio.
The planner's problem

- The planner controls the economy in two ways:
  - accumulates and distributes liquidity
  - reallocates payoffs at date 3

- Formally, the planner chooses:
  - an initial cash balance $m_0$
  - an amount $x_1(\theta_1)$ to distribute in state $\theta_1$ at date 1 and a balance $m_1(\theta_1)$ to carry forward, where
    $$x_1(\theta_1) = m_0 - m_1(\theta_1)$$
  - an amount $x_2(\theta_1, \theta_2)$ to distribute to bankers in state $(\theta_1, \theta_1)$ at date 2 and a balance $m_1(\theta_1, \theta_2)$ to carry forward, where
    $$x_2(\theta_1, \theta_2) = m_1(\theta_1) - m_2(\theta_1, \theta_2)$$
The planner’s solution

- Suppose the planner has $m_1(\theta_1)$ units of cash at date 2 and the state is $(\theta_1, \theta_2)$; the optimal policy is to choose

$$x_2(\theta_1, \theta_2) = \min \{m_1(\theta_1), (1 - \theta_1)\theta_2\}$$

and $m_2(\theta_1, \theta_2) = m_1(\theta_1) - x_2(\theta_1, \theta_2)$

- Suppose the planner has $m_0$ units of cash at date 1 and the state is $\theta_1$; the optimal policy is to choose

$$x_1(\theta_1) = \min \{m_0, \theta_1\}$$

and $m_1(\theta_1) = m_0 - x_1(\theta_1)$

- At date 0, the planner holds $m_0$ units of cash, where $m_0$ satisfies

$$(R - 1) \Pr[\theta_1 + (1 - \theta_1)\theta_2 > m_0] + 1 = \rho$$
The constrained-efficient allocation

**Theorem**

The planner’s optimal strategy is characterized by an array 
$(m_0, m_1(\theta_1), m_2(\theta_1, \theta_2))$ defined by the following conditions:

$m_2(\theta_1, \theta_2) = \max \{m_1(\theta_1) - (1 - \theta_1)\theta_2, 0\}$;

$m_1(\theta_1) = \max \{m_0 - \theta_1, 0\}$

and

$$R \left( 1 - \int_0^{m_0} F_2 \left( \frac{m_0 - \theta_1}{1 - \theta_1} \right) f_1(\theta_1) \, d\theta_1 \right) + 1 = \rho.$$
The market economy

- **Date 0**: a fraction $\alpha$ of bankers choose to consume their cash and $1 - \alpha$ decide to hold it.

- **Date 1**: A fraction $\theta_1$ of bankers receive a demand for payment; assets can be sold on the spot market to raise cash; failure to pay leads to default and liquidation.

- **Date 2**: A fraction $\theta_2$ of bankers receive a demand for payment; assets can be sold on the spot market to raise cash; failure to pay leads to default and liquidation.

- At date 3, solvent bankers receive the returns from the assets they hold; remaining debts are due and paid.
1 - $\alpha$ agents choose to become *liquid* agents; the remainder are *illiquid* agents.

- A fraction $\theta_1$ of agents are hit by a liquidity shock.
- Illiquid agents who receive a shock trade the asset for cash or default.
- Liquid agents who do not receive a shock become either *buyers* or *hoarders*.

- A fraction $\theta_2$ of agents are hit by a liquidity shock.
- Illiquid agents and ‘buyers’ who receive a shock, trade the asset for cash or default.
- Hoarders who do not receive a shock buy assets or hold cash.

- Asset returns are consumed.

$t = 0$  
$t = 1$  
$t = 2$  
$t = 3$
Figure 2: Allocations at dates 0 and 1

\[ (1,0) \]

\( \alpha \) (illiquid)

\[ (1,1) \]

\( 1 - \alpha \) (liquid)

\( \theta_i \) (shock)

\( 1 - \theta_i \) (no shock)

\( 1 - \lambda \) (hoarder)

\( 1 + \lambda \) (buyer)

\( p_i \)
Figure 3a: Allocations at date 2

- **Illiquid**
  - $\theta_2$
  - $1 - \theta_2$

- **Buyer**
  - $\theta_2$
  - $1 - \theta_2$

- **Hoarder**
  - $\theta_2$
  - $1 - \theta_2$

- (max $\{1 - p_2, 0\}, 0$)
- $(1, 0)$
- $(1 + p_1 - p_2, 0)$
- $(1 + p_1, 0)$
- $(1, 0)$
- $(1 + p_2, 0)$
Figure 3b: Allocations at date 2

\[
\begin{align*}
\alpha &\quad \text{(illiquid)} \\
1 - \alpha &\quad \text{(liquid)} \\
\theta_1 &\quad \text{Shock} \\
1 - \theta_1 &\quad \text{No Shock} \\
\theta_2 &\quad \text{Shock} \\
1 - \theta_2 &\quad \text{No Shock} \\
\lambda &\quad \text{Buy} \\
1 - \lambda &\quad \text{Hoard} \\
\theta_2 &\quad \text{Shock} \\
1 - \theta_2 &\quad \text{No Shock} \\
\theta_2 &\quad \text{Shock} \\
1 - \theta_2 &\quad \text{No Shock} \\
\end{align*}
\]

\[
\begin{align*}
\alpha\theta_1 \\
\alpha(1 - \theta_1)\theta_2 \\
\alpha(1 - \theta_1)(1 - \theta_2) \\
(1 - \alpha)\theta_1 \\
(1 - \alpha)(1 - \theta_1)\lambda\theta_2 \\
(1 - \alpha)(1 - \theta_1)(1 - \lambda)\theta_2 \\
(1 - \alpha)(1 - \theta_1)(1 - \lambda)(1 - \theta_2) \\
\end{align*}
\]
Demand and supply for cash at date 2

- Demand for cash comes from
  - the “buyers” hit by a liquidity shock
    \[(1 - \alpha) (1 - \theta_1) \theta_2 \lambda\]
  - and the illiquid bankers hit by a liquidity shock
    \[\alpha (1 - \theta_1) \theta_2\]

- The supply of cash comes
  - from hoarders who do not receive a liquidity shock
    \[(1 - \alpha) (1 - \theta_1) (1 - \theta_2) (1 - \lambda)\]
Market clearing at date 2

- Supply is greater than demand if

\[ (1 - \alpha) (1 - \theta_1) (1 - \theta_2) (1 - \lambda) > (1 - \alpha) (1 - \theta_1) \theta_2 \lambda + \alpha (1 - \theta_1) \theta_2 \]

\[ \iff (1 - \alpha) (1 - \theta_2) (1 - \lambda) > (1 - \alpha) \theta_2 \lambda + \alpha \theta_2 \]

\[ \iff \theta_2 < \theta_2^* = (1 - \alpha) (1 - \lambda) \]

- Demand from “buyers” is greater than supply if

\[ (1 - \alpha) (1 - \theta_1) \theta_2 \lambda > (1 - \alpha) (1 - \theta_1) (1 - \theta_2) (1 - \lambda) \]

\[ \iff \theta_2 \lambda > (1 - \theta_2) (1 - \lambda) \]

\[ \iff \theta_2 > \theta_2^{**} = 1 - \lambda \]
Figure 5C: Different demand and supply regimes as functions of $\theta_2$

(i) $p_2 = \frac{1}{R}$

(ii) $p_2 = 1$

(iii) $p_2 = 1 + p_1$
There are three possible regimes:

1. Supply of cash is *high* relative to demand
   \[ \theta_2 < \theta_2^* \text{ and } p_2 = \frac{1}{R} \]

2. Supply of cash is *intermediate* relative to demand
   \[ \theta_2^* < \theta_2 < \theta_2^{**} \text{ and } p_2 = 1 \]

3. Supply of cash is *low* relative to demand
   \[ \theta_2 > \theta_2^{**} \text{ and } p_2 = 1 + p_1 \]
Buying and hoarding at date 1

- **Buying** is optimal if

\[ p_1(\theta_1) \geq E[p_2(\theta_1, \theta_2) | \theta_1] \]

- **Hoarding** is optimal if

\[ p_1(\theta_1) \leq E[p_2(\theta_1, \theta_2) | \theta_1] \]

Let \( \lambda(\theta_1) \) denote the fraction of liquid bankers who choose to buy assets in state \( \theta_1 \) at date 1. Equilibrium requires

\[ 0 < \lambda(\theta_1) < 1 \]

for every value of \( \theta_1 \)

Hence,

\[ p_1(\theta_1) = E[p_2(\theta_1, \theta_2) | \theta_1] . \]
Market clearing at date 1

- From the equilibrium distribution of $\tilde{p}_2$, we can calculate

$$E [\tilde{p}_2] = F_2 (\theta_2^*) R^{-1} + F_2 (\theta_2^{**}) - F_2 (\theta_2^*) + (1 - F_2 (\theta_2^{**}) (1 + p_1))$$

so the equilibrium condition $p_1 = E [\tilde{p}_2]$ implies that

$$p_1 = \tilde{p} (\lambda) \equiv \frac{1 - F_2 ((1 - \alpha) (1 - \lambda)) (1 - R^{-1})}{F_2 (1 - \lambda)}$$

- By inspection, $\tilde{p} (\lambda)$ is increasing and

$$p (0) < 1 \text{ and } p (1) > 1$$

- Hence, there is a unique value of $\lambda$, call it $\bar{\lambda} \in (0, 1)$, such that $\tilde{p} (\bar{\lambda}) = 1$ and $\tilde{p} (\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$. 
Market clearing at date 1

- If $p_1 < 1$, market-clearing requires

$$
\lambda (\theta_1) (1 - \alpha) (1 - \theta_1) = \alpha \theta_1
$$

or

$$
\lambda (\theta_1) = \frac{\alpha \theta_1}{(1 - \alpha)(1 - \theta_1)}
$$

- The equilibrium value of $\lambda (\theta_1)$ is given by

$$
\lambda (\theta_1) = \min \left\{ \frac{\alpha \theta_1}{(1 - \alpha)(1 - \theta_1)}, \bar{\lambda} \right\}, \text{ for any } \theta_1
$$

- The equilibrium value of $p (\theta_1)$ is given by

$$
p_1 (\theta_1) = \min \left\{ \tilde{p} \left( \frac{\alpha \theta_1}{(1 - \alpha)(1 - \theta_1)} \right), 1 \right\}, \text{ for any } \theta_1
$$
Market clearing at date 0

- In equilibrium at date 0, $0 < \alpha < 1$, which implies that bankers must be indifferent between acquiring liquidity and not acquiring it.

- Agents are indifferent if and only if

$$\int_0^1 p_1 \{1 + (1 - \theta_1)(1 - F_2(\theta_2^{**}))E[\theta_2 | \theta_2 > \theta_2^{**}])\} f_1(\theta_1)d\theta_1 = \frac{\rho}{R}.$$
Equilibrium

An *equilibrium* is described by the endogenous variables $\alpha$, $\lambda(\theta_1)$, $p_1(\theta_1)$, and $p_2(\theta_1, \theta_2)$ satisfying the following conditions:

- at date 2, for every value of $(\theta_1, \theta_2)$, $p_2(\theta_1, \theta_2)$ is the market clearing price, given the values of $\alpha$, $\lambda(\theta_1)$ and $p_1(\theta)$

- at date 1, for every value of $\theta_1$, $\lambda(\theta_1)$ and $p_1(\theta)$ satisfy the market clearing conditions, given the value of $\alpha$

- at date 0, bankers are indifferent between acquiring liquidity and not acquiring it
Suppose that $\alpha = 1$ and that the Bank pursues the socially optimal

At date 2, the market-clearing price is denoted by $p_2 (\theta_1, \theta_2)$ and defined by

$$p_2 (\theta_1, \theta_2) = \begin{cases} 
1 & \text{if } (1 - \theta_1) \theta_2 > \max \{ m_0^* - \theta_1, 0 \} \\
\frac{1}{R-1} & \text{if } (1 - \theta_1) \theta_2 < \max \{ m_0^* - \theta_1, 0 \}
\end{cases}$$

At date 1, the market clearing price is assumed to be

$$p_1 (\theta_1) = \begin{cases} 
1 & \text{if } \theta_1 > m_0^* \\
E [p_2 (\theta_1, \theta_2) | \theta_1] & \text{if } \theta_1 < m_0^*
\end{cases}$$
Optimality I

- An illiquid banker’s payoff is

\[ E [\theta_1 R (1 - p_1 (\theta_1)) + (1 - \theta_1) \theta_2 R (1 - p_2 (\theta_1, \theta_2)) + (1 - \theta_1) (1 - \theta_2) R] = E [R - (\theta_1 + (1 - \theta_1) \theta_2) p_2 (\theta_1, \theta_2) R] \]

- A liquid banker’s payoff is

\[ E [R + (1 - \theta_1) (1 - \theta_2) p_2 (\theta_1, \theta_2) R] - \rho \]

- Then it is optimal to be illiquid if and only if

\[ E [p_2 (\theta_1, \theta_2) R] \leq \rho \]
The first-order condition for the planner’s problem is

\[ R + 1 - R \int_0^{m_0} F_2 \left( \frac{m_0 - \theta_1}{1 - \theta_1} \right) f_1(\theta_1) \, d\theta_1 = \rho. \]

From the definition of \( p_2(\theta_1, \theta_2) \),

\[
E [p_2(\theta_1, \theta_2) R] = R - (R - 1) \int_0^{m_0^*} F_2 \left( \frac{m_0^* - \theta_1}{1 - \theta_1} \right) f_1(\theta_1) \, d\theta_1 \\
\leq R + 1 - R \int_0^{m_0^*} F_2 \left( \frac{m_0^* - \theta_1}{1 - \theta_1} \right) f_1(\theta_1) \, d\theta_1 \\
\leq \rho
\]
The decentralization theorem has an important corollary: the solution to the planner’s problem can be extended to an economy with private information.

Suppose that the aggregate shocks $\theta_1$ and $\theta_2$ are public information, but a banker’s liquidity shock is private information.

The planner chooses an incentive compatible direct mechanism: at each date $t = 1, 2$, bankers announce whether they have received a liquidity shock.

At date $t = 1, 2$, the planner allocates one unit of cash with probability $\mu_1(\theta_1)$ (resp. $\mu_2(\theta_1, \theta_2)$) to a banker who claims to have received a liquidity shock.

In exchange, the banker supplies $p_1(\theta_1)$ (resp. $p_2(\theta_1, \theta_2)$) units of the asset.

The mechanism $(\mu_1(\theta_1), p_1(\theta_1), \mu_2(\theta_1, \theta_2))$ is incentive compatible in the sense that truth telling is an optimal strategy.
Incomplete information

- The equilibrium in which the Lender of Last Resort supplies all the liquidity is equivalent to an incentive compatible direct mechanism

- Set

\[ \mu_1 (\theta_1) = \frac{x_1 (\theta_1)}{\theta_1} \text{ and } \mu_2 (\theta_1, \theta_2) = \frac{x_2 (\theta_1, \theta_2)}{(1 - \theta_1) \theta_2} \]

and let \( p_1 (\theta_1) \) and \( p_2 (\theta_1, \theta_2) \) be the equilibrium prices

- The optimality of the bankers’ behavior implies incentive compatibility

Theorem

*The allocation that solves the planner’s problem can be implemented by an incentive-compatible direct mechanism*
Market regulation I

- Suppose the CB can control $\lambda$ while allowing markets to clear at other dates.
- The optimal level of $\lambda^{soc}$ has the same structure as the equilibrium $\lambda$ but is larger:
  \[ \lambda^{soc} = \min \left\{ \frac{\alpha \theta_1}{(1 - \alpha)(1 - \theta_1)}, \tilde{\lambda} \right\}, \text{ where } \tilde{\lambda} > \bar{\lambda} \]
- It is not optimal to set $\bar{\lambda} = 1$: because bankers are not allowed to make the optimal hoarding decision, the value of holding cash is reduced, other things being equal.
- The CB faces a tradeoff between efficient allocation of aggregate liquidity at date 0 and the amount of aggregate liquidity at date 1.
The diagram illustrates the constrained inefficient region. The axes represent λ and θ, with specific thresholds indicated by θ̅ and θ̃. The dotted lines denote the constrained region, where the parameter space is limited by the given constraints.
Now suppose the CB can only control the quantity of aggregate liquidity \( \alpha \) at date 0 while allowing markets to clear at other dates.

The welfare maximizing value \( \alpha^{soc} \) is smaller than the equilibrium level of \( \alpha \).

*Intuition*: by “envelope theorem” argument, increased aggregate liquidity lowers cost of liquidity at date 2.

It is never optimal to set \( \alpha^{soc} = 0 \) unless \( \rho = 1 \).
A model without hoarding I

- To show that
- Three dates, $t = 0, 1, 2$; liquidity shock $\theta_1$ at date 1; returns at date 3
- $1 - \alpha$ hold cash at date 0, $(1 - \alpha) \theta_1$ supply their own cash needs and $(1 - \alpha) (1 - \theta_1)$ have spare cash to lend:

$$p_1(\theta_1) = \begin{cases} 
1 & \text{if } \theta_1 > 1 - \alpha, \\
R^{-1} & \text{if } \theta_1 < 1 - \alpha.
\end{cases}$$

- The allocation of cash at date 1 is efficient, but the equilibrium allocation is not efficient: $\alpha$ too low because utility of lenders not taken into account
Bankers are indifferent between being liquid or illiquid if $E[p_1] = \frac{\rho}{R}$ or
\[
F_1(1 - \alpha) = \frac{R - \rho}{R - 1}.
\]

The planner’s FOC is
\[
(R - 1)(1 - F_1(m_0)) + 1 = \rho,
\]
or
\[
F_1(m_0) = 1 - \frac{\rho - 1}{R - 1} = \frac{R - \rho}{R - 1}
\]

Thus, $m_0 = 1 - \alpha$ and the equilibrium allocation is constrained efficient
Asset price volatility and fire sales

- When large bankers default at date 2, they raise the price of liquidity to

\[ p_2(\theta_1, \theta_2) = 1 + p_1(\theta_1) \]

creating a fire sale of assets

- The anticipation of this asset price volatility increases both the precautionary and speculative motives for hoarding liquidity

- In fact, if we remove the excess volatility, we can show that inefficient hoarding disappears

- Suppose that the liquidity shock experienced by the bank is the demand for payment of a non-recourse loan and that the collateral for this loan is the initial endowment of one unit of the asset

- When the banker receives a liquidity shock, only one unit of the asset is at risk of being liquidated
Asset price volatility and fire sales II

- The maximum amount that an illiquid banker is willing to pay for cash at date 2 is one unit of the asset, since he will lose only one unit if he defaults.

- Then the equilibrium price at date 2 is given by

\[ p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{w. pr. } F_2((1 - \alpha)(1 - \lambda(\theta_1))), \\ 1 & \text{w. pr. } 1 - F_2((1 - \alpha)(1 - \lambda(\theta_1))). \end{cases} \]

- As before, we can show that market-clearing at date 1 requires

\[ p_1(\theta_1) = E[p_2(\theta_1, \theta_2) \mid \theta_1] \]

- Inefficient hoarding at date 1 implies \( p_1(\theta_1) = 1 \), so \( p_2(\theta_1, \theta_2) = 1 \) with probability one, but this is impossible if there is a positive amount of hoarding.

- So there is no inefficient hoarding in equilibrium.
Incomplete markets

- The fundamental cause of inefficiency in this model is the incompleteness of markets, i.e., the absence of markets for insuring liquidity shocks.

- Under symmetric information, trading contingent claims (to cash and assets) could achieve the first best.

- Under asymmetric information, it may be impossible to improve on the allocation achieved through spot markets.

- Suppose that individual liquidity shocks are private information; then a market mechanism must give bankers incentives to reveal their information truthfully.

- We show that the equilibrium cannot be improved on by the introduction of an incentive compatible market mechanism when there is asymmetric information.
Direct mechanism I

- By the revelation principle, we can restrict our attention to direct mechanisms.
- Let \( \{\alpha, \lambda(\theta_1), p_1(\theta_1), p_2(\theta_1, \theta_2)\} \) be an equilibrium and consider the effect of opening a market for liquidity insurance at date 0.
- At date 0, bankers enter into contracts to deliver or receive liquidity under specified conditions.
- Suppliers acquire one unit of liquidity at date 0; demanders do not.
- At dates \( t = 1, 2 \), each banker is required to report his type, that is, whether or not he has received a liquidity shock.
- Suppliers who report “shock” and demanders who report “no shock” do not trade.
At date 1,

- a supplier who reports “no shock” receives \((-1, \hat{p}_1(\theta_1))\) with probability \(\nu_1(\theta_1)\)
- a demander who reports “shock” receives \((1, -\hat{p}(\theta_1))\) with probability \(\mu_1(\theta_1)\)

At date 2,

- a supplier who reports “no shock” for the second time and has not traded receives \((-1, \hat{p}_2(\theta_1, \theta_2))\) with probability \(\nu_2(\theta_1, \theta_2)\)
- a demander who reports “shock” for the first time receives \((1, -\hat{p}_2(\theta_1, \theta_2))\) with probability \(\mu_2(\theta_1, \theta_2)\)
The impossibility of insurance

- If $\hat{p}_1 (\theta_1) > p_1 (\theta_1)$, a demander who receives a shock will report “no shock” and buy on the spot market; if $\hat{p}_1 (\theta_1) < p_1 (\theta_1)$, a supplier who did receive a shock will report “shock” and sell on the spot market.

- Thus, incentive compatibility at date 1 requires

  $$\hat{p}_1 (\theta_1) = p_1 (\theta_1), \text{ for every } \theta_1$$

- Similarly, incentive compatibility at date 2 requires

  $$\hat{p}_2 (\theta_1, \theta_2) = p_2 (\theta_1, \theta_2), \text{ for every } (\theta_1, \theta_2)$$
Conclusion

- Goodfriend and King argued that it is sufficient to provide adequate liquidity to the system as a whole ...

- ... but should the Central Bank become the sole provider of liquidity?

- Limits of the Lender of Last Resort
  - inflation
  - counterparty risk
  - asset risk
  - moral hazard
  - the unwind problem