Simplest New Keynesian Model

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Outline

• Basic components of a model analysis:
  – What is the best possible allocation of labor and consumption, independent of how these is brought about.
  – How might markets cause this to happen (the ‘invisible hand’).
  – What happens with markets that resemble actual real world markets where prices don’t adjust instantly.

• Do these things in the basic New Keynesian model without capital

• Implications of model for monetary policy:
  – Clarifying the concepts of ‘excess and inadequate aggregate demand’.
  – The Taylor principle and inflation targeting.
  – Cases where ‘overzealous inflation targeting’ can go awry:
    • News shocks and the relationship between monetary policy and stock market volatility
    • The working capital channel and the Taylor principle.
Model

• Household preferences:

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right\}, \]

\[ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iidN(0, \sigma_{\varepsilon}^2) \]
Production

• Final output requires lots of intermediate inputs:
  \[ Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \varepsilon > 1 \]

• Production of intermediate inputs:
  \[ Y_{i,t} = e^{a_i}N_{i,t}, \quad \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \quad \varepsilon_t^a \sim iidN(0, \sigma_a^2) \]

• Constraint on allocation of labor:
  \[ \int_0^1 N_{it} \, di = N_t \]
Efficient Allocation of Total Labor

• Suppose total labor, $N_t$, is fixed.

• What is the best way to allocate $N_t$ among the various activities, $0 \leq i \leq 1$?

• Answer:
  – allocate labor equally across all the activities

$$N_{it} = N_t, \text{ all } i$$
Suppose Labor *Not* Allocated Equally

• Example:

$$N_{it} = \begin{cases} 
2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\
2(1 - \alpha)N_t & i \in \left[\frac{1}{2}, 1\right]
\end{cases}, \quad 0 \leq \alpha \leq 1.$$  

• Note that this is a particular distribution of labor across activities:

$$\int_0^1 N_{it}di = \frac{1}{2}2\alpha N_t + \frac{1}{2}2(1 - \alpha)N_t = N_t$$
Labor Not Allocated Equally, cnt’d

\[ Y_t = \left[ \int_0^1 Y_{i,t}^{\varepsilon-1} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = \left[ \int_0^{\frac{1}{2}} Y_{i,t}^{\varepsilon-1} \, di + \int_{\frac{1}{2}}^{1} Y_{i,t}^{\varepsilon-1} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{\alpha_t} \left[ \int_0^{\frac{1}{2}} N_{i,t}^{\varepsilon-1} \, di + \int_{\frac{1}{2}}^{1} N_{i,t}^{\varepsilon-1} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{\alpha_t} \left[ \int_0^{\frac{1}{2}} (2\alpha N_t)^{\varepsilon-1} \, di + \int_{\frac{1}{2}}^{1} (2(1 - \alpha)N_t)^{\varepsilon-1} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{\alpha_t} N_t \left[ \int_0^{\frac{1}{2}} (2\alpha)^{\varepsilon-1} \, di + \int_{\frac{1}{2}}^{1} (2(1 - \alpha))^\frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{\alpha_t} N_t \frac{1}{2} (2\alpha)^{\varepsilon-1} + \frac{1}{2} (2(1 - \alpha))^\frac{\varepsilon-1}{\varepsilon} \]

\[ = e^{\alpha_t} N_t f(\alpha) \]
Efficient Resource Allocation Means Equal Labor Across All Sectors

\[ f(\alpha) = \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1 - \alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]
Economy with Efficient $N$ Allocation

- Efficiency dictates
  \[ N_{it} = N_t \text{ all } i \]

- So, with efficient production:
  \[ Y_t = e^{a_t} N_t \]

- Resource constraint:
  \[ C_t \leq Y_t \]

- Preferences:
  \[
  E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid,
  \]
Efficient Determination of Labor

- Lagrangian:

\[
\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} - \log C_t + \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} + \lambda_t \left[ e^{a_l} N_t - C_t \right] \right\}
\]

- First order conditions:

\[
u_c(C_t, N_t, \tau_t) = \lambda_t, \quad u_n(C_t, N_t, \tau_t) + \lambda_t e^{a_l} = 0
\]

- or:

\[
u_n + u_c e^{a_l} = 0
\]

marginal cost of labor in consumption units = \(-\frac{du}{dC_t} = \frac{dC_t}{dN_t}\)

\[
\frac{u_n}{u_c} = e^{a_l}
\]

marginal product of labor
Efficient Determination of Labor, cont’d

- Solving the fonc’s:

\[
\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}
\]

\[
C_t \exp(\tau_t)N^\phi_t = e^{a_t}
\]

\[
e^{a_t}N_t \exp(\tau_t)N^\phi_t = e^{a_t}
\]

\[
\Rightarrow N_t = \exp\left(\frac{-\tau_t}{1 + \phi}\right)
\]

\[
\Rightarrow C_t = \exp\left(a_t - \frac{\tau_t}{1 + \phi}\right)
\]

- Note:
  - Labor responds to preference shock, not to tech shock
Response to a Jump in $a$

Production frontier

Indifference curve

Higher $a$

consumption

Leisure, 1-N
Case Where Markets Work Beautifully (triumph of the ‘invisible hand’)

• Give households budget constraints, put them in markets and let them pursue their individual interests.

• Give the production functions to firms and suppose that they seek to maximize profits.

• There is monopoly power....extinguish the effects of that by a subsidy.

• Let markets work perfectly: prices and wages adjust instantly all the time, to clear markets.
Households

• Solve:

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right\},
\]

• Subject to:

Profits, net of government taxes

bonds purchases in \( t \) \quad wage rate \quad profits \quad (real) interest on bonds

\[
C_t + \overbrace{B_{t+1}} \leq \underbrace{w_t N_t}_{\text{wage rate}} + \underbrace{\pi_t}_{\text{profits}} + \underbrace{r_{t-1}}_{\text{(real) interest on bonds}} B_t
\]

• First order conditions:

\[
\frac{-u_{n,t}}{u_{c,t}} = C_t \exp(\tau_t) N_t^{\varphi} = w_t \quad \text{‘marginal cost of working equals marginal benefit’}
\]

\[
u_{c,t} = \beta E_t u_{c,t+1} r_t \quad \text{‘marginal cost of saving equals marginal benefit’}
\]
Final Good Firms

• Final good firms buy $Y_{i,t}, i \in (0, 1)$, at given prices, $P_{i,t}$, to maximize profits:

$$Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

• Subject to

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

• Fonc’s:

$$P_{i,t} = \left( \frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\varepsilon}}$$

$$\Rightarrow Y_{i,t} = P_{i,t}^{\varepsilon} Y_t, 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$
Intermediate Good Firms

• For each $Y_{i,t}$ there is a single producer who is a monopolist in the product market and hires labor, $N_{i,t}$ in competitive labor markets.

• Marginal cost of production:

$$(\text{real) marginal cost} = s_t = \frac{d\text{Cost}}{d\text{worker}} - \frac{d\text{output}}{d\text{worker}} = \left(1 - \frac{\text{subsidy payment to firm}}{\exp(a_t)}\right)w_t$$

• Subsidy will be required to ensure markets work efficiently.
Intermediate Good Firms

Demand curve

\[ Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t \]

Marginal cost, \( S_t \)

Marginal revenue
ith Intermediate Good Firm

- Problem: \( \max_{N_{it}} P_{it} Y_{it} - s_t Y_{it} \)

- Subject to demand for \( Y_{i,t} \): \( Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t \)

- Problem:

\[
\max_{N_{it}} P_{it} P_{i,t}^{-\varepsilon} Y_t - s_t P_{i,t}^{-\varepsilon} Y_t
\]

\( f_{\text{onc}} : (1 - \varepsilon) P_{it}^{-\varepsilon} Y_t + \varepsilon s_t P_{i,t}^{-\varepsilon - 1} Y_t = 0 \)

\( P_{it} = \frac{\varepsilon}{\varepsilon - 1} s_t \) ‘price is markup over marginal cost’

- Note: all prices are the same, so resources allocated efficiently across intermediate good firms.

\( P_{i,t} = P_{j,t} = 1, \) because \( 1 = \int_0^1 P_{i,t}^{1-\varepsilon} \, di \)
Equilibrium

• Pulling things together:

\[ 1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - v)w_t}{\exp(a_t)} \]

- household fonc

\[ = \frac{\varepsilon(1 - v)}{\varepsilon - 1} \frac{-u_{n,t}}{u_{c,t}} \exp(a_t) \]

- if \( \frac{\varepsilon(1-v)}{\varepsilon-1} = 1 \)

\[ = \frac{-u_{n,t}}{u_{c,t}} \exp(a_t). \]

If proper subsidy is provided to monopolists, employment is efficient:

if \( 1 - v = \frac{\varepsilon - 1}{\varepsilon} \), then \( \frac{-u_{n,t}}{u_{c,t}} = \exp(a_t) \)
Equilibrium Allocations

• With efficient subsidy,

\[
\frac{-u_{n,t}}{u_{c,t}} \implies C_t \exp(\tau_t)N_t^\phi = \exp(a_t) \exp(\tau_t)N_t^{1+\phi} = \exp(a_t)
\]

\[
\implies N_t = \exp\left(\frac{-\tau_t}{1 + \phi}\right)
\]

\[
C_t = e^{a_t}N_t = \exp\left(a_t - \frac{\tau_t}{1 + \phi}\right)
\]

• Bond market clearing implies:

\[B_t = 0\] always
Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

\[ u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t \]

\[ \rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\phi}]} \]

\[ = \frac{1}{\beta \exp[E_t(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\phi}) + \frac{1}{2} V]}, \quad V = \sigma_a^2 + \left( \frac{1}{1 + \phi} \right)^2 \sigma_\lambda^2 \]

\[ \log r_t = -\log \beta + E_t \left( \frac{c_{t+1} - c_t}{\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1 + \phi}} \right) + \frac{1}{2} V \]

using assumptions about \( \Delta a_t \) and \( \tau_t \)

\[ \approx -\log \beta + \rho \Delta a_t - \frac{(\lambda-1)\tau_t}{1+\phi} + \frac{1}{2} V \]
Dynamic Properties of the Model

Response to .01 Technology Shock in Period 1

\[(a_t - a_{t-1}) = 0.75(a_{t-1} - a_{t-2}) + \varepsilon_{at}\]

Response to .01 Preference Shock in Period 1

\[\tau_t = 0.5\tau_{t-1} + \varepsilon_{stau}\]

Interest rate

\[\log r_t = -\log \beta + 0.75\Delta a_t\]

\[\log r_t = -\log \beta - (0.5 - 1)\tau_t/(1 + 1)\]
Key Features of Equilibrium Allocations

- Allocations *efficient* (as long as monopoly power neutralized)
- Employment does not respond to technology
  - Improvement in technology raises marginal product of labor and marginal cost of labor by same amount.
- First best consumption not a function of intertemporal considerations
  - Discount rate irrelevant.
  - Anticipated future values of shocks irrelevant.
- Natural rate of interest steers consumption and employment towards their natural levels, ‘as if guided by an invisible hand’.
Introducing Price Setting Frictions (Clarida-Gali-Gertler Model)

- Households maximize:

\[
E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid,
\]

- Subject to:

\[
P_tC_t + B_{t+1} \leq W_tN_t + R_{t-1}B_t + T_t
\]

- Intratemporal first order condition:

\[
C_t \exp(\tau_t)N_t^{\varphi} = \frac{W_t}{P_t}
\]

Profits, net of taxes raised by Government to finance subsidies.
Household Intertemporal FONC

• Condition:

\[ 1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1 + \pi_{t+1}} \]

– or

\[ 1 = \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1 + \pi_{t+1}} \]
\[ = \beta E_t \exp[\log(R_t) - \log(1 + \pi_{t+1}) - \Delta c_{t+1}] \]
\[ \approx \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}], \quad c_t \equiv \log(C_t) \]

– take log of both sides:

\[ 0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1}, \quad r_t = \log(R_t) \]

– or

\[ c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1} \]
NK IS Curve

- Euler equation in two equilibria:

  Actual equilibrium: \( y_t = -[r_t - E_t \pi_{t+1} - rr] + E_t y_{t+1} \)

  Natural equilibrium: \( y_t^* = -[r_t^* - rr] + E_t y_{t+1}^* \)

- Subtract:

  \( x_t = -[r_t - E_t \pi_{t+1} - r_t^*] + E_t x_{t+1} \)
Final Good Firms

- Buy $Y_{i,t}$, $i \in [0, 1]$ at prices $P_{i,t}$ and sell $Y_t$ for $P_t$
- Take all prices as given (competitive)
- Profits:
  \[
P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} \, di
  \]
- Production function:
  \[
  Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \varepsilon > 1,
  \]
- First order condition:
  \[
  Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \quad \rightarrow \quad P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{1}{1-\varepsilon}}
  \]
Intermediate Good Firms

• Each ith good produced by a single monopoly producer.

• Demand curve:

\[ Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \]

• Technology:

\[ Y_{i,t} = \exp(a_t)N_{i,t}, \quad \Delta a_t = \rho \Delta a_{t-1} + \varepsilon^a_t, \]

• Calvo Price-setting Friction

\[ P_{i,t} = \begin{cases} 
\tilde{P}_t & \text{with probability } 1 - \theta \\
P_{i,t-1} & \text{with probability } \theta 
\end{cases} \]
real marginal cost = \[ s_t = \frac{\frac{dCost}{dwor ker}}{\frac{dOutput}{dwor ker}} = \frac{(1 - \nu)W_t/P_t}{\exp(a_t)} \]

= \frac{\varepsilon - 1}{\varepsilon} \text{ in efficient setting}

= \frac{(1 - \nu)C_t \exp(\tau_t)N_t^\phi}{\exp(a_t)}
The Intermediate Firm’s Decisions

- *ith* firm is required to satisfy whatever demand shows up at its posted price.

- Its only real decision is to adjust price whenever the opportunity arises.

- It sets its price as a function of current and expected future marginal cost.
  - Inflation evolves (to first order approximation) as follows:

\[
\pi_t = \frac{(1 - \theta)(1 - \beta\theta)}{\theta} [\hat{s}_t + \beta E_t \hat{s}_{t+1} + \beta^2 E_t \hat{s}_{t+2} + \ldots ]
\]

\[
\hat{s}_t \equiv \frac{s_t - s}{s}, \quad \pi_t \equiv \frac{P_t - P_{t-1}}{P_{t-1}}
\]
NK Phillips Curve

\[ \beta E_t \pi_{t+1} = \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \left[ \beta E_t \hat{s}_{t+1} + \beta^2 E_t \hat{s}_{t+2} + \beta^3 E_t \hat{s}_{t+3} + \ldots \right] \]

\[ \pi_t - \beta E_t \pi_{t+1} = \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \hat{s}_t \]

\[ \hat{s}_t = (1 + \phi) x_t \]

\[ \rightarrow \pi_t = \beta E_t \pi_{t+1} + \kappa x_t, \quad \kappa = \frac{(1 - \theta)(1 - \beta \theta)}{\theta} (1 + \phi) \]
Taylor Rule

- Policy rule

\[ r_t = \alpha r_{t-1} + (1 - \alpha) \left[ r_r + \phi_\pi \pi_t + \phi_x x_t \right] \]

, \( x_t \equiv y_t - y_t^* \).
Equations of Actual Equilibrium
Closed by Adding Policy Rule

\[
\beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \quad \text{(Phillips curve)}
\]

\[-[r_t - E_t \pi_{t+1} - r^*_t] + E_t x_{t+1} - x_t = 0 \quad \text{(IS equation)}
\]

\[\alpha r_{t-1} + (1 - \alpha) \phi \pi \pi_t + (1 - \alpha) \phi_x x_t - r_t = 0 \quad \text{(policy rule)}
\]

\[r^*_t - \rho \Delta a_t - \frac{1}{1 + \phi} (1 - \lambda) \tau_t = 0 \quad \text{(definition of natural rate)}
\]
Solving the Model

\[
\begin{align*}
    s_t &= \left( \begin{array}{c} 
        \Delta a_t \\
        \tau_t
    \end{array} \right) \\
    &= \left[ \begin{array}{cc}
        \rho & 0 \\
        0 & \lambda
    \end{array} \right]\left( \begin{array}{c}
        \Delta a_{t-1} \\
        \tau_{t-1}
    \end{array} \right) + \left( \begin{array}{c}
        \varepsilon_t \\
        \varepsilon_t^r
    \end{array} \right)
\end{align*}
\]

\[
s_t = Ps_{t-1} + \epsilon_t
\]

\[
\begin{align*}
    s_t &= \left[ \begin{array}{cccc}
        \beta & 0 & 0 & 0 \\
        \frac{1}{\sigma} & 1 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0
    \end{array} \right]\left( \begin{array}{c}
        \pi_{t+1} \\
        x_{t+1} \\
        r_{t+1} \\
        r_{t+1}^*
    \end{array} \right) + \left[ \begin{array}{cccc}
        -1 & \kappa & 0 & 0 \\
        0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\
        (1-\alpha)\phi_\pi & (1-\alpha)\phi_x & -1 & 0 \\
        0 & 0 & 0 & 1
    \end{array} \right]\left( \begin{array}{c}
        \pi_t \\
        x_t \\
        r_t \\
        rr_t^*
    \end{array} \right) \\
    &\quad + \left[ \begin{array}{cccc}
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & \alpha & 0 \\
        0 & 0 & 0 & 0
    \end{array} \right]\left( \begin{array}{c}
        \pi_{t-1} \\
        x_{t-1} \\
        r_{t-1} \\
        r_{t-1}^*
    \end{array} \right) + \left( \begin{array}{cc}
        0 & 0 \\
        0 & 0 \\
        0 & 0 \\
        -\sigma\psi\rho & -\frac{1}{\sigma+\phi}(1-\lambda)
    \end{array} \right)\left( \begin{array}{c}
        s_{t+1} \\
        0
    \end{array} \right) + \left( \begin{array}{c}
        0 \\
        0
    \end{array} \right) \\
    &= \left( \begin{array}{c}
        \pi_{t+1} \\
        x_{t+1} \\
        r_{t+1} \\
        rr_{t+1}^*
    \end{array} \right) + \left( \begin{array}{c}
        \pi_t \\
        x_t \\
        r_t \\
        rr_t^*
    \end{array} \right)
\end{align*}
\]

\[
E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0
\]
Solving the Model

\[ E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \]

\[ s_t - Ps_{t-1} - \epsilon_t = 0. \]

• **Solution:**

\[ z_t = Az_{t-1} + Bs_t \]

• **As before:**

\[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \]

\[ F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0 \]
\( \phi_x = 0, \phi_\pi = 1.5, \beta = 0.99, \varphi = 1, \rho = 0.2, \theta = 0.75, \alpha = 0, \delta = 0.2, \lambda = 0.5. \)
Dynamic Response to a Preference Shock

- Inflation
- Output gap
- Nominal rate

- Natural real rate
- Actual nominal rate

- Natural output
- Actual output

- Natural employment
- Actual employment